CROSSING SYMMETRY

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We've looked at some properties of Mandelstam variables, so it's time to see how they are used in field theory. We're considering an interaction in which two particles enter, then interact in some undefined potential, then two, possibly different, particles leave the system. The general diagram is as in Fig. 1.

The Mandelstam variables are defined as

$$s = (p_1 + p_2)^2 \tag{1}$$

$$t = (p_1 + p_3)^2 \tag{2}$$

$$u = (p_1 + p_4)^2 \tag{3}$$

These variables are displayed on a Mandelstam-Kibble plot, as in Fig. 2.

Consider first an interaction in which the timeline in Fig. 1 is taken to run from right to left. This corresponds to a reaction of the form

$$1 + 2 \rightarrow \overline{3} + \overline{4} \tag{4}$$

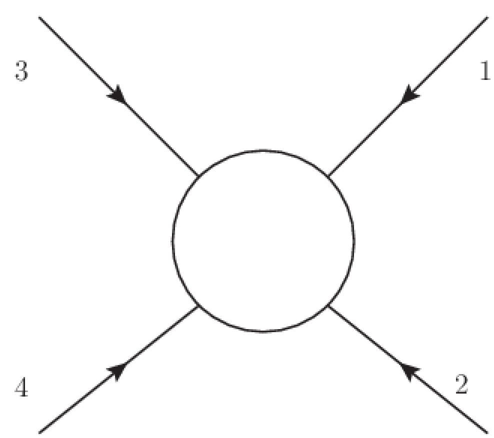

FIGURE 1. Momenta of 4 particles used in Mandelstam variables.

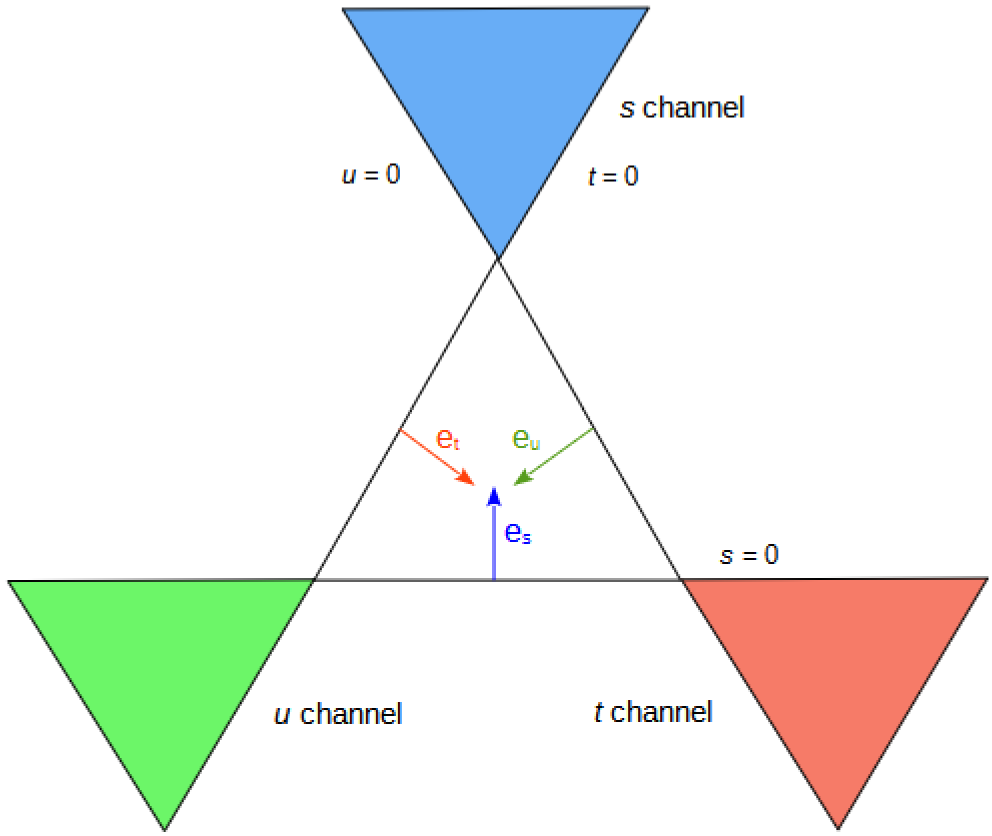

FIGURE 2. Mandelstam-Kibble plot.

This uses our usual convention in which a line with an arrow pointing towards the interaction is a particle, and an arrow pointing away from the interaction is an antiparticle. For simplicity, we'll take the masses of all 4 particles to be equal to m, and we'll consider things in the centre of momentum frame. Also, we're using Coleman's convention here in which the directions of all 4 momenta are taken to be pointing towards the centre. This means that, depending on the interaction, some of the momenta will be negative, since the sum of the incoming momenta must equal the sum of the outgoing momenta.

For this interaction, the total energy is given by

$$(p_1+p_2)^2 = s \geq (2m)^2 \tag{5}$$

The other two Mandelstam variables represent momentum transfers. Since we're taking all four momenta to be pointing inwards, momenta p_3 and p_4 in 4 are negative. Thus

$$(p_1+p_3)^2 = t \tag{6}$$

is the direct momentum transfer. Because the masses are all equal and we're in the centre of momentum frame, the energies of all four particles will also be equal. Thus

$$t = (p_1+p_3)^2 = [(E,\boldsymbol{p}_1)-(E,\boldsymbol{p}_3)]^2 \tag{7}$$

$$= -(\boldsymbol{p}_1-\boldsymbol{p}_3)^2 \leq 0 \tag{8}$$

Recall that the minus sign in the last line comes from the relativistic metric. Conservation of momentum ensures that

$$|\boldsymbol{p}_1| = |\boldsymbol{p}_3| \tag{9}$$

although their directions may be different.

By the same argument, $u = (p_1+p_4)^2$ is the cross momentum transfer, so we have

$$u = -(\boldsymbol{p}_1-\boldsymbol{p}_4)^2 \leq 0 \tag{10}$$

Thus for interaction 4, we see that the physical region of the Mandelstam-Kibble plot is where $s \geq 4m^2$, $t \leq 0$ and $u \leq 0$. From our earlier post we know that the height of the triangle is $M \equiv 4m^2$ so the physical region for interaction 4 is the blue triangle in Fig. 2. This type of interaction is called an *s-channel* interaction.

Now suppose we look at Fig. 1 with a timeline reading from top to bottom. In this case, particles 1 and 3 are incoming and particles 2 and 4 are outgoing, so we have the interaction

$$1+3 \rightarrow \overline{2}+\overline{4} \tag{11}$$

In this case, momenta p_1 and p_3 are positive and p_2 and p_4 are negative. Thus here we have

$$t = (p_1+p_3)^2 \geq 4m^2 \tag{12}$$

$$s = -(\boldsymbol{p}_1-\boldsymbol{p}_2)^2 \leq 0 \tag{13}$$

$$u = -(\boldsymbol{p}_1-\boldsymbol{p}_4)^2 \leq 0 \tag{14}$$

By the same arguments are for the s-channel, we see that in this case, the physical region is the red triangle in Fig. 2. This is a *t-channel* interaction.

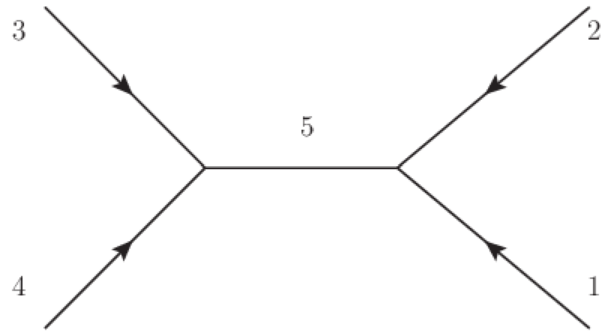

FIGURE 3. Exchange of m_5 particle.

Finally, if we read Fig. 1 from the bottom up, we have

$$1+4\rightarrow\overline{2}+\overline{3} \tag{15}$$

and the relevant Mandelstam variables become

$$u=(p_1+p_4)^2\geq 4m^2 \tag{16}$$

$$s=-(\boldsymbol{p}_1-\boldsymbol{p}_2)^2\leq 0 \tag{17}$$

$$t=-(\boldsymbol{p}_1-\boldsymbol{p}_3)^2\leq 0 \tag{18}$$

This corresponds to the green triangle in Fig. 2, and is called a *u-channel* interaction.

As an example of how all this is useful, Coleman considers an interaction in which the blob in Fig. 1 becomes the exchange of a single particle with mass m_5, as in Fig. 3.

Reading right to left, and applying the Feynman rule for this interaction, we get a term in the scattering amplitude proportional to

$$\frac{1}{(p_1+p_2)^2-m_5^2} \tag{19}$$

As we discussed in a previous post, this term is a pole in the energy, with an analog in first order perturbation theory from nonrelativistic quantum theory. In terms of the Mandelstam variables for this interaction, this term is the same as

$$\frac{1}{s-m_5^2} \tag{20}$$

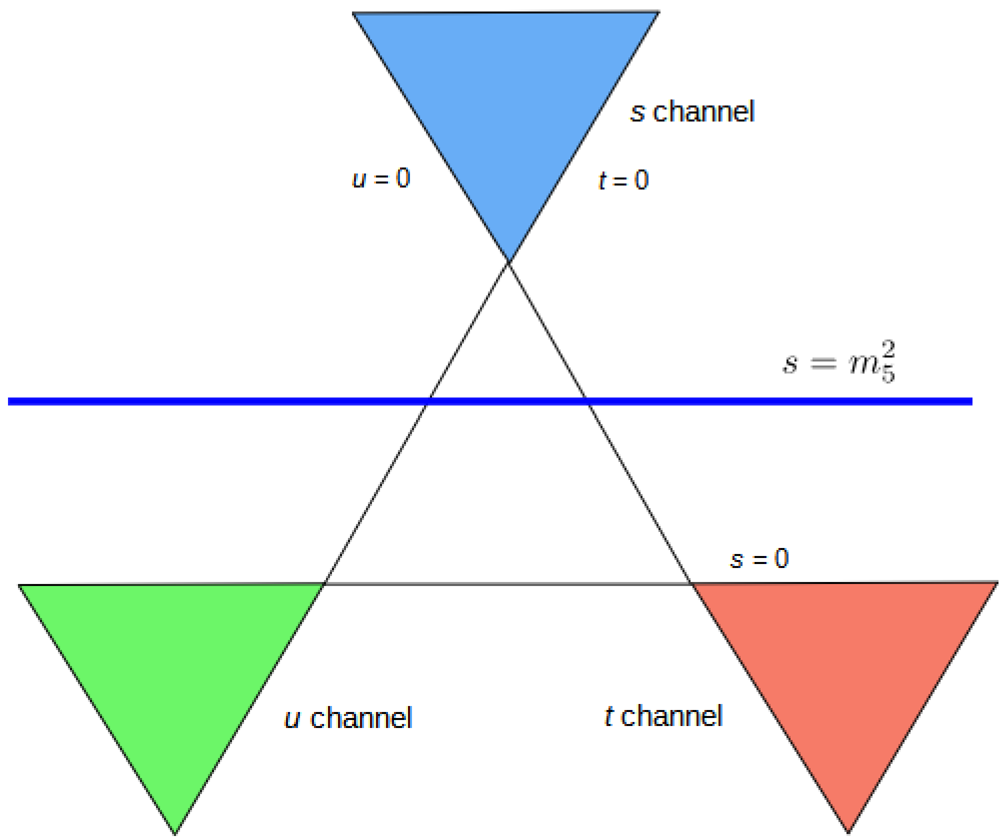

FIGURE 4. Pole at $s = m_5^2$.

Since the two incoming particles must have enough energy to create the virtual m_5 particle, we must have $s > m_5^2$. However, on a Mandelstam-Kibble plot, this pole is drawn as a horizontal line, as shown by the blue line in Fig. 4.

The pole is drawn as a line rather than a single point as would be the case for an ordinary function of a single complex variable because here we have a function of two complex variables p_1 and p_2.

From the above discussion, we see that the s variable in this process, which corresponds to the energy, becomes the direct momentum transfer in the *t*-channel, and then the cross momentum transfer in the *u*-channel.

Coleman says that all three processes are connected by analytic continuation. I think what he's getting at is that it's possible to move the s variable smoothly from one channel to another by using analytic continuation. As we found in the earlier discussion, analytic continuation is a process whereby a function with one or more poles can be extended to cover the entire complex plane except for these poles by using a sequence of Taylor expansions about various points. Each Taylor expansion has a domain of convergence that extends out to the nearest pole.

As we see in Fig. 4, the pole in a Mandelstam-Kibble plot appears as a line that cuts across the entire plane, so we might think there's no way we could use analytic continuation to extend the function 20 to the other side of this line, since if we attempt analytic continuation from any point within the s-channel, the domain of convergence will always hit the barrier at $s = m_5^2$. However, this line exists only in the real plane. Thus if we allow s to take on complex values (which are non-physical, because the quantity $(p_1 + p_2)^2$ in the physical realm is always real) we *can* use analytic continuation to effectively 'go around' the line by dipping into the complex space behind it.

If you recall our earlier example, we considered the function

$$f(z) = \frac{1}{1-z} \tag{21}$$

This has a pole on the real axis at $z = 1$. If we restricted ourselves to the real axis, it's impossible to analytically continue the function to reach points where $z > 1$. To do it, we had to use a Taylor expansion about some point off the real axis. We have a similar situation here, where there is a barrier in the real plane that can be circumvented only by dipping into complex variables.

This symmetry that analytic continuation allows us to see is called *crossing symmetry*.