CURRENTS FROM SPACETIME TRANSLATIONS AND THE ENERGY-MOMENTUM TENSOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 11 August 2021.

In Section 5.5, Coleman applies Noether's theorem in classical field theory to the case of spacetime translation. The transformation is applied to the fields, so we have

$$\phi^{a}(x)\rightarrow\phi^{a}(x+\lambda e) \tag{1}$$

where λ is the parameter giving the size of the translation and e is a constant 4-vector in spacetime giving the direction of the translation. It's important to remember that x and e are 4-vectors, not scalars.

We wish to find the current given by

$$J^{\mu}\equiv\pi_{a}^{\mu}D\phi^{a}-F^{\mu} \tag{2}$$

with the quantities defined as

$$D\phi^{a}\equiv\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0} \tag{3}$$

$$D\mathscr{L}=\partial_{\mu}F^{\mu} \tag{4}$$

$$\pi_{a}^{\mu}=\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi^{a})} \tag{5}$$

From 1 we have

$$D\phi^{a}=e_{\rho}\partial^{\rho}\phi^{a}(x) \tag{6}$$

Since we're considering infinitesimal translations, all the derivative terms $D\ldots$ depend linearly on the components of the vector e, so we will expect the currents J^{μ} to be some linear combination of e_{ρ}. The most general such expression is

$$J^{\mu}=e_{\rho}T^{\rho\mu} \tag{7}$$

where $T^{\rho\mu}$ is, at this stage, nothing more than a general 4×4 matrix.

We saw earlier that

$$\partial_\mu J^\mu = 0 \tag{8}$$

so from 7 this gives us (since the components e_ρ are arbitrary)

$$\partial_\mu T^{\rho\mu} = 0 \tag{9}$$

The quantity $T^{\rho\mu}$ is known as the *canonical energy-momentum tensor*.

At this stage, it's useful to compare this result with that for classical particle theory, where the system was translated in space only by $\boldsymbol{x} \to \boldsymbol{x} + \lambda \boldsymbol{e}$. In that case, Noether's theorem resulted in a global conservation law for the quantity

$$Q = \boldsymbol{e} \cdot \sum_r m_r \dot{\boldsymbol{x}}^r \tag{10}$$

$$= \boldsymbol{e} \cdot \boldsymbol{p} \tag{11}$$

where $\boldsymbol{p}$ is the total momentum. Remember that in particle mechanics, the conservation laws are global, applying to the system as a whole, but in field theory, the currents J^μ are *densities*, and we get both a local and a global conservation law.

Returning to field theory, we observe that, so far, the only condition we have on $T^{\rho\mu}$ is 9, and therefore $T^{\rho\mu}$ is not unique. This is because we can add the divergence of some antisymmetric object $A^{\rho\mu\lambda}$, where the antisymmetry applies to the last two indices:

$$A^{\rho\mu\lambda} = -A^{\rho\lambda\mu} \tag{12}$$

We can therefore use a modified energy-momentum tensor of form

$$\theta^{\rho\mu} = T^{\rho\mu} + \partial_\lambda A^{\rho\mu\lambda} \tag{13}$$

because the divergence with respect to μ of the last term is zero:

$$\partial_\mu \partial_\lambda A^{\rho\mu\lambda} = -\partial_\lambda \partial_\mu A^{\rho\lambda\mu} = -\partial_\mu \partial_\lambda A^{\rho\mu\lambda} = 0 \tag{14}$$

The first step follows because of 12 and the fact that the order of the derivatives doesn't matter. The second step follows from swapping the dummy indices $\lambda \leftrightarrow \mu$, and the last step follows because the first and third terms are negatives of each other, so must be 0.

The tensor $T^{\rho\mu}$ contains 16 quantities. We can get some insight into what these components mean physically by applying the integral form of the conservation law 8. From 7, suppose we take

$$e = (1,0,0,0) \tag{15}$$

That is, e represents a translation in time only. Then we have

$$J^{\mu}=T^{0\mu} \tag{16}$$

and from 8 we have

$$\partial_0 T^{00}+\partial_i T^{0i}=0 \tag{17}$$

Integrating this over a volume V with surface S and using Gauss's theorem, we get

$$\frac{d}{dt}\int_V d^3\boldsymbol{x}\;T^{00} = -\int_V d^3\boldsymbol{x}\;\partial_i T^{0i} \tag{18}$$

$$= -\int_S d^2\boldsymbol{x}\;n_i T^{0i} \tag{19}$$

where $\boldsymbol{n}$ is a unit normal to the surface.

In this case, the 'charge'-like quantity is the integral of T^{00} and the 'current'-like quantity is T^{0i}. We see that the change in the integral of T^{00} is given by the integral of the flow of this 'stuff' across the surface. In the particle theory, invariance under time translation led to conservation of energy. In the classical field theory, we therefore interpret T^{00} as the mass-energy density (since we're doing a relativistic theory, we have the usual mass-energy equivalence), and its space integral is equal and opposite to the surface integral on the RHS of 19. Therefore, the quantities T^{0i} are the components of a 3-vector representing the flow of energy across the surface. The physical interpretation of 19 is therefore that the rate of change of the mass-energy content of a volume V is given by the rate at which energy flows across the bounding surface S.

We have a similar interpretation for the other components, although in this case, it is momentum density and momentum flow. For example, if

$$e=(0,1,0,0) \tag{20}$$

so that the system is translated along the x^1 axis only, we get

$$\frac{d}{dt}\int_V d^3\boldsymbol{x}\;T^{10} = -\int_V d^3\boldsymbol{x}\;\partial_i T^{1i} \tag{21}$$

$$= -\int_S d^2\boldsymbol{x}\;n_i T^{1i} \tag{22}$$

In this case, the conserved 'stuff' is the total momentum in the x^1 direction, given by $\int_V d^3\boldsymbol{x}\;T^{10}$, and the quantities T^{1i} represent the flow, or current, of x^1 momentum in each of the three coordinate directions.

Thus the T^{00} component is the mass-energy density, and the T^{i0} component is the density of the ith component of 3-momentum.

To complete the derivation, Coleman derives an explicit form for J^μ using 2 (details in his eqns 5.46 to 5.50) with the result that

$$J^\mu = \pi_a^\mu e_\rho \partial^\rho \phi^a - g^{\mu\rho} e_\rho \mathscr{L} \tag{23}$$

where $g^{\mu\rho}$ is the metric tensor. Comparing this with 7 and using the fact that e_ρ is arbitrary so we can factor it out, we get an explicit form for the energy-momentum tensor:

$$\boxed{T^{\rho\mu} = \pi_a^\mu \partial^\rho \phi^a - g^{\mu\rho} \mathscr{L}} \tag{24}$$

Although all this was derived for *classical* field theory, it turns out to be valid for quantum field theory as well. Coleman goes through the derivation for the scalar quantum field

$$\phi(\boldsymbol{x},t) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^\dagger e^{ip\cdot x}\right) \tag{25}$$

In eqns 5.54 through 5.58 he shows that, with a bit of mathematical licence (juggling an infinite delta function) or by using normal ordering, we can derive the conserved momentum for the quantum field, giving

$$\boldsymbol{P} = \int d^3\boldsymbol{p} \, \left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}\right) \boldsymbol{p} \tag{26}$$

PINGBACKS

Pingback: Noether's theorem and Lorentz transformations