DIRAC EQUATION - 4 SOLUTION VECTORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 24 August 2023.

The Dirac equation in relativistic quantum mechanics can be written as

$$(i\gamma^{\mu}\partial_{\mu}-mI)|\psi\rangle=0 \tag{1}$$

When written out in its matrix components, this equation is actually 4 differential equations.

$$(i\partial_0-m)|\psi\rangle_1+i\partial_3|\psi\rangle_3+(i\partial_1+\partial_2)|\psi\rangle_4=0 \tag{2}$$
$$(i\partial_0-m)|\psi\rangle_2+(i\partial_1-\partial_2)|\psi\rangle_3-i\partial_3|\psi\rangle_4=0 \tag{3}$$
$$-i\partial_3|\psi\rangle_1-(i\partial_1+\partial_2)|\psi\rangle_2-(i\partial_0+m)|\psi\rangle_3=0 \tag{4}$$
$$-i(\partial_1+i\partial_2)|\psi\rangle_1+i\partial_3|\psi\rangle_2-(i\partial_0+m)|\psi\rangle_4=0 \tag{5}$$

Remember that $|\psi\rangle$ is a 4-d column vector in spinor space rather than a single function, so that the subscript index j in $|\psi\rangle_j$ indicates which component in spinor space we're dealing with. These equations have four solutions denoted by $\left|\psi^{(n)}\right\rangle$ for $n=1,2,3,4$. Note that each $\left|\psi^{(n)}\right\rangle$ is a full 4-component vector in spinor space; that is, the superscript (n) indicates which complete vector we're dealing with. Thus $\left|\psi^{(n)}\right\rangle_j$ is the jth component of the nth vector.

We can write the 4 PDEs as a matrix eigenvalue equation by moving the terms involving m to the RHS and factoring out an i from the terms remaining on the LHS:

$$i\left[\begin{array}{cccc}\partial_0 & 0 & \partial_3 & \partial_1-i\partial_2\\ 0 & \partial_0 & \partial_1+i\partial_2 & -\partial_3\\ -\partial_3 & -\partial_1+i\partial_2 & -\partial_0 & 0\\ -\partial_1-i\partial_2 & \partial_3 & 0 & -\partial_0\end{array}\right]\left[\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\right]=m\left[\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{array}\right] \tag{6}$$

We'll now look at the four solutions $\left|\psi^{(n)}\right\rangle$ and verify that they satisfy 6. First, we have

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}1\\0\\\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\end{bmatrix}e^{-ipx} \equiv u_1 e^{-ipx} \tag{7}$$

where u_1 is defined by this equation as the constant $\sqrt{\frac{E+m}{2m}}$ multiplied by the 4-d spinor factor. Remember that px is a 4-vector product:

$$px = p^\mu x_\mu = Et - \mathbf{p}\cdot\mathbf{x} \tag{8}$$

The derivatives in 6 are all with respect to spacetime variables, so act only on e^{-ipx}; the spinor components are constants with respect to these derivatives. The first row in 6 is therefore

$$\begin{aligned}
i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-iE+0+\frac{p^3}{E+m}\left(ip^3\right)+\frac{p^1+ip^2}{E+m}\left(ip^1+p^2\right)\right] &= \quad (9)\\
-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{\mathbf{p}^2}{E+m}\right] &= \quad (10)\\
-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{E^2-m^2}{E+m}\right] &= \quad (11)\\
-\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-E+\frac{(E+m)(E-m)}{E+m}\right] &= \sqrt{\frac{E+m}{2m}}e^{-ipx}m \quad (12)\\
&= m\psi_1 \quad (13)
\end{aligned}$$

Thus the first row of 6 is verified. The other 3 rows can be verified similarly. For row 2:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[0+0+\frac{p^3}{E+m}\left(ip^1-p^2\right)+\frac{p^1+ip^2}{E+m}\left(-ip^3\right)\right] = 0 = m\psi_2 \tag{14}$$

For row 3:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[-ip^3+0+\frac{p^3}{E+m}(iE)+0\right]=i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{-ip^3(E+m)+ip^3E}{E+m}\right] \tag{15}$$

$$=\sqrt{\frac{E+m}{2m}}e^{-ipx}m\frac{p^3}{E+m} \tag{16}$$

$$=m\psi_3 \tag{17}$$

And for row 4:

$$i\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\left(-ip^1+p^2\right)+0+0+\frac{p^1+ip^2}{E+m}iE\right]= \tag{18}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\left(p^1+ip^2\right)-\frac{p^1+ip^2}{E+m}E\right]= \tag{19}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\left[\frac{\left(p^1+ip^2\right)(E+m)-\left(p^1+ip^2\right)E}{E+m}\right]= \tag{20}$$

$$\sqrt{\frac{E+m}{2m}}e^{-ipx}\frac{p^1+ip^2}{E+m}m=m\psi_4 \tag{21}$$

The other 3 solutions are

$$\left|\psi^{(2)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}0\\1\\\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\end{bmatrix}e^{-ipx}\equiv u_2e^{-ipx} \tag{22}$$

$$\left|\psi^{(3)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^3}{E+m}\\\frac{p^1+ip^2}{E+m}\\1\\0\end{bmatrix}e^{ipx}\equiv v_2e^{ipx} \tag{23}$$

$$\left|\psi^{(4)}\right\rangle=\sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{p^1-ip^2}{E+m}\\-\frac{p^3}{E+m}\\0\\1\end{bmatrix}e^{ipx}\equiv v_1e^{ipx} \tag{24}$$

If you really want to, you can verify that these 3 vectors satisfy 6 by grinding through the calculations as above. One point worth noting is that the constant $\sqrt{\frac{E+m}{2m}}$ that multiplies all the solutions could be any other constant and still satisfy 6 (since the constant just cancels off both sides). It's chosen to be $\sqrt{\frac{E+m}{2m}}$ to make later calculations easier.

PINGBACKS

Pingback: Dirac equation - inner products of spinors
Pingback: Dirac equation - orthogonality of solutions
Pingback: Dirac equation - adjoint solutions
Pingback: Dirac equation - positive probabilities and negative energies
Pingback: Dirac equation - spin of a particle at rest
Pingback: Dirac equation - spin of a moving particle
Pingback: Dirac equation - spinors near the speed of light
Pingback: Dirac equation - non-relativistic limit
Pingback: Helicity operator in the Dirac equation
Pingback: Dirac equation in relativistic quantum mechanics - summary
Pingback: Anticommutators, creation and annihilation operators in the Dirac equation
Pingback: Outer product of spinors
Pingback: Electromagnetic interaction form of the Dirac equation