DIRAC EQUATION - ADJOINT SOLUTIONS

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The four solutions of the Dirac equation in relativistic quantum mechanics are

$$\left|\psi^{(1)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \tag{1}$$

$$\left|\psi^{(2)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \tag{2}$$

$$\left|\psi^{(3)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \tag{3}$$

$$\left|\psi^{(4)}\right\rangle = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \tag{4}$$

We've seen that these solutions are mutually orthogonal by taking the inner product of each solution with the complex conjugate transpose of another solution. You might think that we could derive a sort of conjugate transpose version of the Dirac equation by using the 'bra' versions of the solutions, but in fact it seems that it is more usual to define an *adjoint* of each solution by taking the conjugate transpose and then post-multiplying it by the matrix γ^0. That is, we define the adjoint solutions $\left\langle\bar{\psi}^{(n)}\right|$ by

$$\left\langle\bar{\psi}^{(n)}\right| \equiv \left\langle\psi^{(n)}\right|\gamma^0 \tag{5}$$

with

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6}$$

We can write out the four adjoints explicitly by taking the conjugate transpose and doing the matrix multiplication. We get

$$\left\langle \bar{\psi}^{(1)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} 1 & 0 & \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} \end{array} \right] e^{ipx}\gamma^0 \tag{7}$$

$$= \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} 1 & 0 & -\frac{p^3}{E+m} & -\frac{p^1-ip^2}{E+m} \end{array} \right] e^{ipx} \tag{8}$$

$$\left\langle \bar{\psi}^{(2)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} 0 & 1 & -\frac{p^1+ip^2}{E+m} & \frac{p^3}{E+m} \end{array} \right] e^{ipx} \tag{9}$$

$$\left\langle \bar{\psi}^{(3)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} \frac{p^3}{E+m} & \frac{p^1-ip^2}{E+m} & -1 & 0 \end{array} \right] e^{-ipx} \tag{10}$$

$$\left\langle \bar{\psi}^{(4)} \right| = \sqrt{\frac{E+m}{2m}} \left[\begin{array}{cccc} \frac{p^1+ip^2}{E+m} & -\frac{p^3}{E+m} & 0 & -1 \end{array} \right] e^{-ipx} \tag{11}$$

Note that post-multiplying by γ^0 just changes the sign of the last two elements in $\left\langle \psi^{(n)} \right|$.

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