

DIRAC EQUATION - MATRIX PROPERTIES

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In our earlier look at the Dirac equation, we simply stated the equation and then derived a few properties of its components. It seems that this is pretty much the way Dirac himself derived it. He was seeking a relativistic generalization of the Schrödinger equation, so he wanted his equation to have the form

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \quad (1)$$

for some Hamiltonian H . The original Schrödinger equation defined H (for a free particle) as $H = -\frac{\nabla^2}{2m}$, so this non-relativistic equation had a first-order time derivative and a second order spatial derivative. The Klein-Gordon equation contains second-order derivatives in both time and space.

The problem Dirac faced was therefore how to define H to make it relativistic and at the same time, retain the first order time derivative. In special relativity, energy, mass and momentum are related by

$$E^2 = p^2 + m^2 \quad (2)$$

If we use the operator form for the momentum: $p^i = -i\partial^i$, then attempting to use $E = \sqrt{p^2 + m^2}$ as the Hamiltonian operator runs into the problem that the momentum operator is inside a square root. Dirac solved this problem by proposing that

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \quad (3)$$

where $\boldsymbol{\alpha}$ is a 3-d vector of matrices and β is a single (scalar) matrix. We can then require that H^2 gives the energy squared as specified in 2, which in turn imposes conditions on the matrices $\boldsymbol{\alpha}$ and β . We saw in the earlier post that this leads to the conditions that all four matrices have zero trace, have ± 1 eigenvalues, and have even dimension, with the minimum dimension being 4×4 . The matrices also satisfy anticommutation relations:

$$\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0 \text{ if } i \neq j \quad (4)$$

$$\alpha_i^2 = \beta^2 = I \quad (5)$$

Dirac and Pauli found that the smallest matrices that satisfy all these conditions are

$$\alpha_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \quad (6)$$

$$\alpha_2 = \begin{bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{bmatrix} \quad (7)$$

$$\alpha_3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix} \quad (8)$$

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \quad (9)$$

where all blank matrix elements are zero. These can be written in a more condensed form by using the Pauli spin matrices from non-relativistic quantum mechanics

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (10)$$

We get

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \quad (11)$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad (12)$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \quad (13)$$

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (14)$$

Here, each entry in each matrix is a 2×2 submatrix given by 10.

From their definitions and the fact that the Pauli matrices are hermitian, all 4 matrices have zero trace and are hermitian (that is, they equal their complex conjugate transpose). The eigenvalues of β are ± 1 since the matrix is diagonal. The eigenvalues of the α_i can be found in the usual way. For α_1 we get

$$\det(\alpha_1 - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \quad (15)$$

$$= -\lambda [-\lambda(\lambda^2) - (-\lambda)] - (\lambda^2 - 1) \quad (16)$$

$$= \lambda^4 - 2\lambda^2 + 1 \quad (17)$$

$$= (\lambda^2 - 1)^2 \quad (18)$$

The roots of this equation are ± 1 (twice each). The eigenvalues of α_2 and α_3 also turn out to be ± 1 as can be verified if you grind through the calculations. The conditions 4 and 5 can be verified by direct multiplication (although this is kind of tedious).

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