

DIRAC EQUATION - NON-RELATIVISTIC LIMIT

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Post date: 26 August 2023.

The solutions to the Dirac equation consist of a 4-element column spinor and a spacetime component:

$$|\psi^{(1)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (1)$$

$$|\psi^{(2)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \end{bmatrix} e^{-ipx} \equiv u_2 e^{-ipx} \quad (2)$$

$$|\psi^{(3)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ipx} \equiv v_2 e^{ipx} \quad (3)$$

$$|\psi^{(4)}\rangle = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{p^1-ip^2}{E+m} \\ -\frac{p^3}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ipx} \equiv v_1 e^{ipx} \quad (4)$$

In the non-relativistic limit, the relative velocity of the particle satisfies $v \ll 1$, which means that the momentum components all satisfy $p^j \ll E \approx m$. Thus $\sqrt{\frac{E+m}{2m}} \approx 1$ and the solutions reduce to

$$|\psi^{(1)}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (5)$$

$$|\psi^{(2)}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (6)$$

$$|\psi^{(3)}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{ipx} \quad (7)$$

$$|\psi^{(4)}\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{ipx} \quad (8)$$

Comparing this with the free particle solutions to the non-relativistic Schrödinger equation, we see that the $e^{\pm ipx} = e^{\pm Et} e^{\mp \mathbf{P} \cdot \mathbf{x}}$ factor is just what we'd get in that case. The first two components of the spinors in $|\psi^{(1)}\rangle$ and $|\psi^{(2)}\rangle$ also correspond to the eigenstates in the non-relativistic spin $\frac{1}{2}$ theory.

In the 4-d case, we can operate on these solutions with the 4-d spin operator Σ_z to get

$$\Sigma_z |\psi^{(1)}\rangle = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (9)$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (10)$$

$$= \frac{1}{2} |\psi^{(1)}\rangle \quad (11)$$

$$\Sigma_z |\psi^{(2)}\rangle = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (12)$$

$$= -\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-ipx} \quad (13)$$

$$= -\frac{1}{2} |\psi^{(2)}\rangle \quad (14)$$

Similarly, we get

$$\Sigma_z |\psi^{(3)}\rangle = \frac{1}{2} |\psi^{(3)}\rangle \quad (15)$$

$$\Sigma_z |\psi^{(4)}\rangle = -\frac{1}{2} |\psi^{(4)}\rangle \quad (16)$$

These are the same results that we get by applying the Pauli spin matrices to the 2-d spin space spinors in the non-relativistic theory.