

DIRAC EQUATION

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The Klein-Gordon equation is an early attempt at a relativistic quantum theory, but it contains a second-order time derivative which leads to probability not being conserved over time. Dirac proposed another equation that attempts to solve this problem for particles of spin 1/2. The Dirac equation is essentially a modification of the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi_a(x) = [-i\hbar c (\alpha^j)_{ab} \partial_j + mc^2 (\beta)_{ab}] \psi_b(x) \quad (1)$$

Here, ψ is now a vector in spin space with components ψ_a . The objects β and α^j (for $j = 1, 2, 3$) are square matrices (where the subscript ab indicates the component of the matrix being considered), also in spin space, and repeated indices are summed over spatial coordinates only. [We won't worry about *how* Dirac arrived at this equation for now; we'll just accept it and see where it leads.]

To make this equation formally equivalent to the Schrödinger equation, the hamiltonian operator H on the RHS must now be a matrix. We can also use the definition of the momentum operator $P_j = -i\hbar \partial_j$ to get

$$H_{ab} = cP_j (\alpha^j)_{ab} + mc^2 (\beta)_{ab} \quad (2)$$

This might not look much like the relativistic energy:

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad (3)$$

but if we square 2 (remembering that matrix products need not commute), we have

$$(H^2)_{ab} = c^2 P_j P_k (\alpha^j \alpha^k)_{ab} + mc^3 P_j (\alpha^j \beta + \beta \alpha^j)_{ab} + m^2 c^4 (\beta^2)_{ab} \quad (4)$$

We can define the anticommutator as

$$\{A, B\} \equiv AB + BA \quad (5)$$

We can write the first term on the RHS of 4 as

$$c^2 P_j P_k (\alpha^j \alpha^k)_{ab} = \frac{1}{2} c^2 P_j P_k \{ \alpha^j, \alpha^k \}_{ab} \quad (6)$$

which works because we're summing over j and k , so

$$P_j P_k \alpha^j \alpha^k = P_j P_k \alpha^k \alpha^j \quad (7)$$

even though the various α matrices need not commute (or anticommute) with each other.

We get

$$(H^2)_{ab} = \frac{1}{2} c^2 P_j P_k \{ \alpha^j, \alpha^k \}_{ab} + m c^3 P_j \{ \alpha^j, \beta \}_{ab} + m^2 c^4 (\beta^2)_{ab} \quad (8)$$

In order to make this equal to E^2 , we need the matrices α^j and β to satisfy the conditions:

$$\{ \alpha^j, \alpha^k \}_{ab} = 2 \delta^{jk} \delta_{ab} \quad (9)$$

$$\{ \alpha^j, \beta \} = 0 \quad (10)$$

$$(\beta^2)_{ab} = \delta_{ab} \quad (11)$$

The first condition requires the anticommutator of α^j and α^k to be zero unless $j = k$, in which case the anticommutator gives the identity matrix. Remember that the superscripts j and k specify which *matrix* we're talking about, while the subscripts ab indicate the *component* of the matrix. The conditions aren't derived; rather they are imposed to make the energy come out right. With these conditions, we have

$$(H^2)_{ab} = (\mathbf{P}^2 c^2 + m^2 c^4) \delta_{ab} \quad (12)$$

which gives the correct operator for the square of the energy.

The question arises as to what these matrices α^j and β are. One candidate is the set of three Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (13)$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (14)$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (15)$$

By direct calculation, we see that $\{ \sigma^i, \sigma^j \} = 2 \delta^{ij}$. For example

$$\{\sigma_x, \sigma_y\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (16)$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad (17)$$

$$= 0 \quad (18)$$

$$\{\sigma_x, \sigma_x\} = 2\sigma_x^2 \quad (19)$$

$$= 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (20)$$

and so on. However, in order to satisfy 10, we need to find a single matrix that anticommutes with all 3 spin matrices. We require

$$\{\sigma_x, \beta\} = \begin{bmatrix} \beta_{21} & \beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix} + \begin{bmatrix} \beta_{12} & \beta_{11} \\ \beta_{22} & \beta_{21} \end{bmatrix} \quad (21)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (22)$$

This gives

$$\beta_{12} = -\beta_{21} \equiv \gamma \quad (23)$$

$$\beta_{11} = -\beta_{22} \equiv \epsilon \quad (24)$$

We then get

$$\{\sigma_z, \beta\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \epsilon & \gamma \\ -\gamma & -\epsilon \end{bmatrix} + \begin{bmatrix} \epsilon & \gamma \\ -\gamma & -\epsilon \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} \epsilon & \gamma \\ \gamma & \epsilon \end{bmatrix} + \begin{bmatrix} \epsilon & -\gamma \\ -\gamma & \epsilon \end{bmatrix} \quad (26)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (27)$$

This gives

$$\epsilon = 0 \quad (28)$$

So finally

$$\{\sigma_y, \beta\} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (29)$$

$$= \begin{bmatrix} i\gamma & 0 \\ 0 & i\gamma \end{bmatrix} + \begin{bmatrix} i\gamma & 0 \\ 0 & i\gamma \end{bmatrix} \quad (30)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (31)$$

So $\gamma = 0$ resulting in $(\beta)_{ab} = 0$. Thus there is no non-zero matrix β that anticommutes with all 3 of the Pauli spin matrices.

So what *can* we say about the Dirac matrices? From 11, we see that the eigenvalues of $\beta^2 = I$ are all 1, so the eigenvalues of β must be ± 1 .

The trace (sum of the diagonal elements) of a matrix is equal to the sum of its eigenvalues (theorem from matrix algebra). To find the trace of β , we can use the anticommutators 9 and 10, together with another theorem from matrix algebra which states that $\text{tr}(AB) = \text{tr}(BA)$ for any square matrices A and B of the same order.

$$\text{tr}(\alpha_1^2 \beta) = \text{tr}(\alpha_1 (\alpha_1 \beta)) \quad (32)$$

$$= \text{tr}((\alpha_1 \beta) \alpha_1) \quad (33)$$

However, from 10, $\alpha_1 \beta = -\beta \alpha_1$ and from 9, $\alpha_1^2 = I$ (the identity matrix), so

$$\text{tr}(\alpha_1^2 \beta) = \text{tr}(\beta) \quad (34)$$

$$= \text{tr}((\alpha_1 \beta) \alpha_1) \quad (35)$$

$$= -\text{tr}((\beta \alpha_1) \alpha_1) \quad (36)$$

$$= -\text{tr}(\beta \alpha_1^2) \quad (37)$$

$$= -\text{tr}(\beta) \quad (38)$$

Hence $\text{tr}(\beta) = -\text{tr}(\beta) = 0$, so β must have an equal number of $+1$ and -1 eigenvalues. In other words, β must be even dimensional, so the smallest size is 4×4 .

We can also find the trace of α^j by starting with $\text{tr}(\alpha^j \beta^2)$ and following through the same steps as above (using $\beta^2 = I$) to show that $\text{tr}(\alpha^j) = -\text{tr}(\alpha^j) = 0$.

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