# DYSON'S FORMULA AND TIME ORDERING

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The differential equation that needs to be solved to obtain the evolution operator $U_I$ in the interaction picture is

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right)=H_I\left(t\right)U_I\left(t,t'\right) \tag{1}$$

where $H_I$ is the interaction Hamiltonian. In most cases, $H_I$ is not a one-dimensional function, so the simple solution

$$U_I\left(t,t'\right)=\exp\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{2}$$

won't work. The problem is that, due to the Baker-Campbell-Hausdorff formula, the exponential of an integral of an operator contains commutators of the operator at different times and, since in general, $H_I$ doesn't commute with itself at different times, we can't just do the integral and then take the exponential.

The problem was solved, in a formal way (not terribly conducive to calculation), by Freeman Dyson in 1949. The solution is *Dyson's formula*:

$$\boxed{U_I\left(t,t'\right)=T\exp\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right)} \tag{3}$$

where the $T$ in front of the exponential is the *time-ordering* symbol. It is interpreted as ordering all terms in decreasing order of time, from left to right. As Coleman puts it, 'left-ist is latest' or 'later on the left'. The time ordering procedure is always to be done first, before any products of terms.

To see what this means when applied to 3, we need to express this formula so that it contains products of terms, which means we have to expand it in a power series. This is done in Coleman's eqn 7.37. The first two terms are

$$U_I\left(t,t'\right)=T\left(1-i\int_{t'}^{t}dt_1\;H_I\left(t_1\right)+\ldots\right) \tag{4}$$

Since these two terms each consist of only a single factor, time ordering has no effect. The effect first becomes visible on the next term:

$$U_I(t,t') = 1 - i\int_{t'}^{t} dt_1\, H_I(t_1) + \frac{(-i)^2}{2!} T\left[\int_{t'}^{t}\int_{t'}^{t} dt_1 dt_2 H_I(t_1) H_I(t_2)\right] + \ldots \tag{5}$$

The double integral contains portions where $t_1 < t_2$ and also portions where $t_1 > t_2$, since both integration variables extend over the same range. The effect of the time ordering can be broken down into two terms:

$$T\left[\int_{t'}^{t}\int_{t'}^{t} dt_1 dt_2 H_I(t_1) H_I(t_2)\right] = \int_{t'}^{t} dt_2 \left[H_I(t_2)\int_{t'}^{t_2} dt_1 H_I(t_1)\right] + \int_{t'}^{t} dt_1 \left[H_I(t_1)\int_{t'}^{t_1} dt_2 H_I(t_2)\right] \tag{6}$$

In the first term on the RHS, $t_2$ is always later than $t_1$, while the opposite is true in the second term. To visualize this integral, think of an ordinary 2-dim integral over the unit square in the $xy$ plane. The double integral $\int_0^1\int_0^1 dx\, dy \ldots$ can be split into two parts:

$$\int_0^1\int_0^1 dx\, dy \ldots \int_0^1 dy \int_0^y dx \ldots + \int_0^1 dx \int_0^x dy \ldots \tag{7}$$

The first integral integrates over $x$ first, from the line $x = 0$ (the $y$ axis) to the line $x = y$ (the 45° diagonal). This integral extends over the upper right-angled triangle obtained by bisecting the square along its lower-left to upper-right diagonal (that is, along the line $y = x$). Thus this integral covers only half the square, and over this area, $y$ is always greater than $x$. The second integral integrates over $y$ first, from the line $y = 0$ (the $x$ axis) to the line $y = x$ (the same 45° diagonal). This covers the lower right-angled triangle obtained from the same bisection of the square as before, and over this area, $x$ is always greater than $y$. Thus the sum of the two integrals covers the entire square.

In fact, since $t_1$ and $t_2$ in 6 are just dummy integration variables, these two terms are the same, so we could write

$$T\left[\int_{t'}^{t}\int_{t'}^{t} dt_1 dt_2 H_I(t_1) H_I(t_2)\right] = 2\int_{t'}^{t} dt_2 \left[H_I(t_2)\int_{t'}^{t_2} dt_1 H_I(t_1)\right] \tag{8}$$

Higher order terms in the expansion can be worked out the same way.

That this formula provides a solution to 1 can be verified directly. Taking the derivative of both sides with respect to $t$ (not $t'$), we have

$$i\frac{\partial}{\partial t} U_I(t,t') = iT\exp\left(-iH_I(t)\int_{t'}^{t} dt'' H_I(t'')\right) \tag{9}$$

Now provided $t > t'$, the time ordering will always place the $H_I(t)$ term on the left, so it can be taken outside the time-ordered term, giving

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right) = iT\exp\left(\left(-iH_I(t)\right)\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right)\right) \tag{10}$$

$$= i\left(-iH_I(t)\right)T\exp\left(-iH_I(t)\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{11}$$

$$= H_I(t)\,T\exp\left(-iH_I(t)\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{12}$$

which agrees with 1.

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