

ELECTROMAGNETIC LORENTZ INVARIANT

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Post date: 29 Jul 2023.

Since the action integral in a relativistic theory must be a Lorentz invariant, the Lagrangian must be a Lorentz scalar. For the free electromagnetic field, the Lagrangian is usually taken to be

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \quad (2)$$

The sign convention in L&B's book is different from that used by other authors.

With this sign convention, we can see by direct calculation that the components of \mathbf{E} and \mathbf{B} are

$$E_i = F_{0i} = -F_{i0} \quad (3)$$

$$B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk} \quad (4)$$

Latin indices i, j, k represent 1, 2 and 3, while Greek indices represent 0, 1, 2, 3.

where ε_{ijk} is the completely antisymmetric Levi-Civita tensor, and repeated indices are summed, as usual.

Another Lorentz invariant quantity that can be constructed from $F_{\mu\nu}$ is

$$\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \quad (5)$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor where each Greek index takes the values 0, 1, 2, 3. The quantity represented by 5 is therefore a Lorentz invariant. To see what it is in terms of \mathbf{E} and \mathbf{B} , we can use 3. In order for a component of $\varepsilon^{\alpha\beta\gamma\delta}$ to be non-zero, all the indices must be different. Consider first the case $\alpha = 0$. The other three indices can then only be some permutation of i, j, k . We have for this component of the sum in 5

$$\varepsilon^{0ijk} F_{0i} F_{jk} = E_i \varepsilon^{ijk} F_{jk} \quad (6)$$

$$= -2E_i B_i \quad (7)$$

$$= -2\mathbf{E} \cdot \mathbf{B} \quad (8)$$

Now suppose $\beta = 0$. We have for this term

$$\varepsilon^{i0jk} F_{i0} F_{jk} = -\varepsilon^{0ijk} (-F_{0i}) F_{jk} \quad (9)$$

$$= \varepsilon^{0ijk} F_{0i} F_{jk} \quad (10)$$

$$= -2\mathbf{E} \cdot \mathbf{B} \quad (11)$$

In the first line, we swapped the indices $i \leftrightarrow 0$ in both ε^{i0jk} and F_{i0} and used the fact that both these objects are antisymmetric.

Now consider $\gamma = 0$. We have

$$\varepsilon^{ij0k} F_{ij} F_{0k} = \varepsilon^{0ijk} F_{ij} F_{0k} \quad (12)$$

$$= -2B_k E_k \quad (13)$$

$$= -2\mathbf{E} \cdot \mathbf{B} \quad (14)$$

In the first line, we did two swaps to move the 0 index in ε^{ij0k} to ε^{0ijk} . As each swap introduces a factor of -1 , they cancel out.

Finally, consider $\delta = 0$. We have

$$\varepsilon^{ijk0} F_{ij} F_{k0} = \varepsilon^{0ijk} F_{ij} F_{0k} \quad (15)$$

$$= -2\mathbf{E} \cdot \mathbf{B} \quad (16)$$

where in the first line, we did 3 swaps to move the 0 from ε^{ijk0} to ε^{0ijk} , and one swap to move F_{k0} to F_{0k} . Each of these 4 swaps introduces a factor of -1 so the net effect is to multiply by $(-1)^4 = 1$.

Adding everything up we have

$$\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} = -8\mathbf{E} \cdot \mathbf{B} \quad (17)$$

Thus $\mathbf{E} \cdot \mathbf{B}$ is a Lorentz invariant.