ELECTROMAGNETIC INTERACTION FORM OF THE DIRAC EQUATION

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The Lagrangian for the electromagnetic field with sources is

$$\mathcal{L}=-\frac{1}{2}\left(\partial_{\nu}A_{\mu}\right)\left(\partial^{\nu}A^{\mu}\right)+ej^{\mu}A_{\mu} \tag{1}$$

As it stands, this Lagrangian applies to photons and their interaction with electric charge and current, as given by j^{μ}. However, the charge and current are due to electrons, which are governed by the Dirac equation, so ultimately we'd like to link the photon Lagrangian with that for electrons.

To do this, we first recognize that j^{μ} represents a density of charge and current, so it should be proportional to the probability current density for electrons. This probability current density is given by

$$j^{\mu}=\bar{\psi}\gamma^{\mu}\psi \tag{2}$$

where ψ is the solution of the Dirac equation for spin-$\frac{1}{2}$ particles, and $\overline{\psi}$ is the adjoint, given by

$$\bar{\psi}=\psi^{\dagger}\gamma^{0} \tag{3}$$

Thus we can couple Maxwell's and Dirac's equations by inserting 2 into 1:

$$\mathcal{L}=-\frac{1}{2}\left(\partial_{\nu}A_{\mu}\right)\left(\partial^{\nu}A^{\mu}\right)+e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \tag{4}$$

This Lagrangian's first term represents free photons and its second term represents interactions between photons and electrons. To get a complete Lagrangian, we must also add in a term representing free electrons. We introduced this originally for the quantum field theory of the free Dirac equation, but the same Lagrangian applies in the case of relativistic quantum mechanics, so we can quote it here.

$$\mathcal{L}_{0}^{1/2}=\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi \tag{5}$$

where the subscript 0 indicates a free field (no sources) and the superscript 1/2 indicates spin $\frac{1}{2}$ (not a square root). We can add this term to 4 to get the full interaction Lagrangian for photons, electrons and their interactions:

$$\mathcal{L}^{1/2,1}=-\frac{1}{2}\left(\partial_{\nu}A_{\mu}\right)\left(\partial^{\nu}A^{\mu}\right)+\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi+e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \tag{6}$$

We can now use the Euler-Lagrange equation to get the full Dirac equation including interactions. The Euler-Lagrange equation is

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}^{n}}\right)-\frac{\partial\mathcal{L}}{\partial\phi^{n}}=0 \tag{7}$$

where the ϕ^{n} represent the various fields (A^{μ}, ψ and $\bar{\psi}$). We've already applied 7 to 4 to get Maxwell's equations with sources. The addition of the free electron term $\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi$ does not affect this derivation, since it doesn't contain A_{μ}. To get the full Dirac equation, we apply 7 to 6 with $\phi^{n}=\bar{\psi}$. We have

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}^{1/2,1}}{\partial\left(\partial\bar{\psi}_{,\mu}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\bar{\psi}}=-\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi-e\gamma^{\mu}\psi A_{\mu}=0 \tag{8}$$

where the first term on the LHS is zero, since 6 doesn't contain any derivatives of $\bar{\psi}$. Thus the full Dirac equation is

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=-e\gamma^{\mu}\psi A_{\mu} \tag{9}$$

We can get the adjoint Dirac equation by applying 7 to 6 with $\phi^{n}=\psi$. We have

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}^{1/2,1}}{\partial\left(\partial\psi_{,\mu}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\psi}=\left(i\partial_{\mu}\bar{\psi}\right)\gamma^{\mu}+m\bar{\psi}-e\bar{\psi}\gamma^{\mu}A_{\mu}=0 \tag{10}$$

Rearranging, we have the adjoint Dirac equation

$$\left(i\partial_{\mu}\bar{\psi}\right)\gamma^{\mu}+m\bar{\psi}=e\bar{\psi}\gamma^{\mu}A_{\mu} \tag{11}$$

This matches the free adjoint Dirac equation when $A_{\mu}=0$, that is, when there is no interaction with photons.

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