ELECTROMAGNETISM LAGRANGIAN WITH SOURCES

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Maxwell's equations for electromagnetism with source charges and currents are, in 4d form:

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}-\partial^{\mu}\left(\partial_{\nu}A^{\nu}\right)=-ej^{\mu} \tag{1}$$

where the potential A^{μ} is

$$A^{\mu}\left(x\right)=\left[\begin{array}{c}\boldsymbol{\Phi}\left(x\right)\\ A^{1}\left(x\right)\\ A^{2}\left(x\right)\\ A^{3}\left(x\right)\end{array}\right] \tag{2}$$

and the current j^{μ} is

$$-ej^{\mu}=\left[\begin{array}{c}\rho\\ j^{1}\\ j^{2}\\ j^{3}\end{array}\right] \tag{3}$$

where $-e$ is the electron charge.

If we use the Lorenz gauge

$$\partial_{\nu}A^{\nu}=0 \tag{4}$$

1 becomes

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}=-ej^{\mu} \tag{5}$$

All this applies to classical (that is, non-quantum) electromagnetism, but when we make the transition to quantum field theory, we'll see that we can use the same equations, but with the fields interpreted in terms of creation and annihilation operators.

As usual, we would like a Lagrangian density that, when fed into the Euler-Lagrange equations, gives us 5 back again. The Euler-Lagrange equations in our case are

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\right)-\frac{\partial\mathcal{L}}{\partial A_\mu}=0 \tag{6}$$

The easiest way to do things is just to state the Lagrangian (from here on, we're working with the Lagrangian *density*, but to be concise I'll refer to it just as the Lagrangian), and then show that applying 6 gives us 5. We have

$$\mathcal{L}=-\frac{1}{2}\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right)+ej^\mu A_\mu \tag{7}$$

In order to apply 6, we need to write 7 with its indices moved to the lower location. We can do this using the metric

$$g^{\mu\nu}=\text{diag}\left(1,-1,-1,-1\right) \tag{8}$$

We have

$$\mathcal{L}=-\frac{1}{2}\left(\partial_\nu A_\mu\right)\left(g^{\nu\alpha}g^{\mu\beta}\partial_\alpha A_\beta\right)+ej^\mu A_\mu \tag{9}$$

The derivative $\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}$ can be evaluated using the product rule on the first term of 9. We have

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}=-\frac{1}{2}\left[\frac{\partial\left(\partial_\nu A_\mu\right)}{\partial\left(\partial_\nu A_\mu\right)}\left(g^{\nu\alpha}g^{\mu\beta}\partial_\alpha A_\beta\right)+\left(\partial_\nu A_\mu\right)\frac{\partial\left(g^{\nu\alpha}g^{\mu\beta}\partial_\alpha A_\beta\right)}{\partial\left(\partial_\nu A_\mu\right)}\right] \tag{10}$$

The derivative in the second term selects the term with $\alpha=\nu$ and $\beta=\mu$ from the numerator. That is,

$$\frac{\partial\left(g^{\nu\alpha}g^{\mu\beta}\partial_\alpha A_\beta\right)}{\partial\left(\partial_\nu A_\mu\right)}=g^{\nu\nu}g^{\mu\mu}\quad\text{(no sum over }\mu,\nu\text{)} \tag{11}$$

This effectively raises the indices on the $\partial_\nu A_\mu$ in the second term of 10. That is (again, with no sum over μ,ν):

$$\partial_\nu A_\mu g^{\nu\nu}g^{\mu\mu}=\partial^\nu A^\mu \tag{12}$$

so we get

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}=-\frac{1}{2}\left[\left(g^{\nu\alpha}g^{\mu\beta}\partial_\alpha A_\beta\right)+\partial^\nu A^\mu\right] \tag{13}$$

$$=-\frac{1}{2}\left[\partial^\nu A^\mu+\partial^\nu A^\mu\right] \tag{14}$$

$$=-\partial^\nu A^\mu \tag{15}$$

For the second term in 6, we have

$$\frac{\partial\mathcal{L}}{\partial A_{\mu}}=ej^{\mu} \tag{16}$$

Putting it all together, we have

$$\partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\nu}A_{\mu}\right)}\right)-\frac{\partial\mathcal{L}}{\partial A_{\mu}}=-\partial_{\nu}\partial^{\nu}A^{\mu}-ej^{\mu}=0 \tag{17}$$

or

$$\partial_{\nu}\partial^{\nu}A^{\mu}=-ej^{\mu} \tag{18}$$

which is 5, as required.

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