ELECTRON-ELECTRON SCATTERING IN SECOND ORDER S MATRIX

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In Chapter 8, Klauber works out the second order S matrix element for electron-positron scattering in detail. Here, we'll examine a similar case, that of electron-electron scattering, sometimes known as Møller scattering. At second order, there are two possible Feynman diagrams for this process, as in Fig. 1.

In the diagrams, time increases upwards.

The Hamiltonian density is

$$\mathcal{H}_I^I = -e\bar{\psi}\not{A}\psi \tag{1}$$

The scattering process involves the destruction of two incoming electrons, the transmission of a virtual photon, and the subsequent creation of two outgoing electrons. The relevant portions of the operators in 1 are

$$\begin{aligned} \psi^+ &\to \text{destroy electron } e^- \\ \bar{\psi}^- &\to \text{create electron } e^- \end{aligned} \tag{2}$$

We'll consider first the left hand diagram in Fig. 1. We need to destroy an electron with momentum p_1 at location x_2 and also an electron with momentum p_2 at location x_1. We then exchange a virtual photon and create

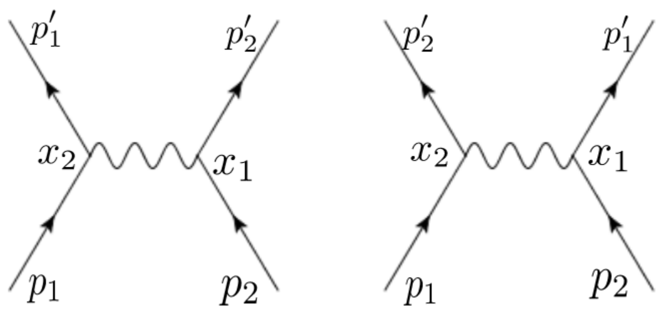

FIGURE 1. Feynman diagrams for electron-electron scattering.

outgoing electrons at x_2 and x_1. The relevant second order term in the expansion of the S matrix is

$$S_{M1}^{(2)}=-\frac{e^{2}}{2}\int d^{4}x_{1}d^{4}x_{2}\;N\left\{\left(\bar{\psi}_{1'}^{-}\not{A}\psi_{1}^{+}\right)_{x_{2}}\left(\bar{\psi}_{2'}^{-}\not{A}\psi_{2}^{+}\right)_{x_{1}}\right\} \tag{3}$$

The subscripts $1,1'$ and so on indicate the momentum of the associated particle, and subscripts x_1,x_2 indicate the vertex at which these terms are evaluated.

According to Wick's theorem, we need to normal-order 3. The terms with the superscript + represent destruction, so we must place these on the right, introducing a minus sign for each swap since we're dealing with fermions.

In what follows, it will be most convenient to write out the matrix indices for the various terms, including the spinors and gamma matrices. So 3 becomes

$$S_{M1}^{(2)}=-\frac{e^{2}}{2}\int d^{4}x_{1}d^{4}x_{2}\;N\left\{\bar{\psi}_{1',\alpha}^{-}\left(x_{2}\right)\gamma_{\alpha\beta}^{\mu}A_{\mu}\psi_{1,\beta}^{+}\left(x_{2}\right)\bar{\psi}_{2',\delta}^{-}\left(x_{1}\right)\gamma_{\delta\epsilon}^{\nu}A_{\nu}\psi_{2,\epsilon}^{+}\left(x_{1}\right)\right\} \tag{4}$$

Here there are sums over all Greek indices. The diagram on the LHS of Fig. 1 has a duplicate with x_1 and x_2 interchanged. This will give the same probability of the interaction, so we double $S_{M1}^{(2)}$ in what follows.

We can replace the contraction with the Feynman propagator for photon fields, as described in Klauber's Chapter 5. We have

$$A_{\mu}\left(x_{2}\right)A_{\nu}\left(x_{1}\right)=iD_{F\mu\nu}\left(x_{1}-x_{2}\right) \tag{5}$$

Doing the normal ordering introduces a minus sign when we swap $\bar{\psi}_{2',\delta}^{-}\left(x_{1}\right)$ with $\psi_{1,\beta}^{+}\left(x_{2}\right)$ so we have

$$S_{M1}^{(2)}=e^{2}\int d^{4}x_{1}d^{4}x_{2}iD_{F\mu\nu}\left(x_{1}-x_{2}\right)\bar{\psi}_{1',\alpha}^{-}\left(x_{2}\right)\gamma_{\alpha\beta}^{\mu}\bar{\psi}_{2',\delta}^{-}\left(x_{1}\right)\psi_{1,\beta}^{+}\left(x_{2}\right)\gamma_{\delta\epsilon}^{\nu}A_{\nu}\psi_{2,\epsilon}^{+}\left(x_{1}\right) \tag{6}$$

We now substitute in the explicit forms of the fermion operators:

$$\psi = \sum_{r=1}^{2} \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[c_r(\mathbf{p})\, u_r(\mathbf{p})\, e^{-ipx} + d_r^{\dagger}(\mathbf{p})\, v_r(\mathbf{p})\, e^{ipx} \right] \tag{7}$$

$$\equiv \psi^+ + \psi^- \tag{8}$$

$$\bar{\psi} = \sum_{r=1}^{2} \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[d_r(\mathbf{p})\, \bar{v}_r(\mathbf{p})\, e^{-ipx} + c_r^{\dagger}(\mathbf{p})\, \bar{u}_r(\mathbf{p})\, e^{ipx} \right] \tag{9}$$

$$\equiv \bar{\psi}^+ + \bar{\psi}^- \tag{10}$$

When $S_{M1}^{(2)}$ acts on the initial state $\left|e_{\boldsymbol{p}_1 r_1} e_{\boldsymbol{p}_2 r_2}\right\rangle$ all terms in the sum will give zero except for those that match the momenta and spins of the incoming electrons. These operators will destroy the incoming electrons, leaving the vacuum state. Thus we get

$$S_{M1}^{(2)} \left|e_{\boldsymbol{p}_1 r_1} e_{\boldsymbol{p}_2 r_2}\right\rangle = e^2 \int d^4x_1 d^4x_2 i D_{F\mu\nu}(x_1 - x_2)\, \bar{\psi}_{1',\alpha}^{-}(x_2)\, \gamma_{\alpha\beta}^{\mu} \bar{\psi}_{2',\delta}^{-}(x_1) \times \frac{m}{V\sqrt{E_{\boldsymbol{p}_1} E_{\boldsymbol{p}_2}}} u_{r_1,\beta}(\boldsymbol{p}_1)\, e^{-ip_1 x_2} u_{r_2,\epsilon}(\boldsymbol{p}_2)\, \gamma_{\delta\epsilon}^{\nu} |0\rangle \tag{11}$$

The propagator has the form

$$D_{F\mu\nu} = \frac{-ig_{\mu\nu}}{(2\pi)^4} \int \frac{e^{-ik(x_1 - x_2)}}{k^2 + i\varepsilon} d^4k \tag{12}$$

Inserting this into 11 we have

$$S_{M1}^{(2)} = -ie^2 \frac{g_{\mu\nu}}{(2\pi)^4} \int d^4x_1 d^4x_2 \left[\int \frac{e^{-ik(x_1 - x_2)}}{k^2 + i\varepsilon} d^4k \right] \bar{\psi}_{1',\alpha}^{-}(x_2)\, \gamma_{\alpha\beta}^{\mu} \bar{\psi}_{2',\delta}^{-}(x_1) \times \frac{m}{V\sqrt{E_{\boldsymbol{p}_1} E_{\boldsymbol{p}_2}}} u_{r_1,\beta}(\boldsymbol{p}_1)\, e^{-ip_1 x_2} u_{r_2,\epsilon}(\boldsymbol{p}_2)\, e^{-ip_2 x_1} \gamma_{\delta\epsilon}^{\nu} \tag{13}$$

We wish to find the matrix element for a transition to the outgoing state with electrons with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$, so when we expand the operators $\bar{\psi}_{1',\alpha}^{-}(x_2)\, \gamma_{\alpha\beta}^{\mu} \bar{\psi}_{2',\delta}^{-}(x_1)$, only those terms that create electrons with the correct momenta will survive. This will result in the replacement:

$$\bar{\psi}_{1',\alpha}^{-}(x_2)\, \gamma_{\alpha\beta}^{\mu} \bar{\psi}_{2',\delta}^{-}(x_1) \rightarrow \frac{m}{V\sqrt{E_{\boldsymbol{p}_1'} E_{\boldsymbol{p}_2'}}} \bar{u}_{r_{1'},\alpha}\left(\boldsymbol{p}_1'\right) e^{ip_1' x_2} \gamma_{\alpha\beta}^{\mu} \bar{u}_{r_2',\delta}\left(\boldsymbol{p}_2'\right) e^{ip_2' x_1} \tag{14}$$

Inserting this into 13 we have

$$
\begin{aligned}
S_{M1}^{(2)} = -ie^2 \frac{g_{\mu\nu}}{(2\pi)^4} \int d^4x_1 d^4x_2 \left[\int \frac{e^{-ik(x_1-x_2)}}{k^2+i\varepsilon} d^4k \right] \times \\
\frac{m}{V\sqrt{E_{\boldsymbol{p}_1'} E_{\boldsymbol{p}_2'}}} \bar{u}_{r_{1'},\alpha} \left(\boldsymbol{p}_1'\right) e^{ip_1'x_2} \gamma^{\mu}_{\alpha\beta} \bar{u}_{r_2',\delta} \left(\boldsymbol{p}_2'\right) e^{ip_2'x_1} \times \\
\frac{m}{V\sqrt{E_{\boldsymbol{p}_1} E_{\boldsymbol{p}_2}}} u_{r_1,\beta} \left(\boldsymbol{p}_1\right) e^{-ip_1x_2} u_{r_2,\epsilon} \left(\boldsymbol{p}_2\right) e^{-ip_2x_1} \gamma^{\nu}_{\delta\epsilon}
\end{aligned} \tag{15}
$$

Since the only places where the x_is appear are the exponentials, we can do the integral over x_2 to get

$$
\int d^4x_2 e^{i\left(k+p_1'-p_1\right)x_2} = (2\pi)^4 \delta^{(4)} \left(k+p_1'-p_1\right) \tag{16}
$$

This delta function indicates conservation of momentum at the vertex x_2. This allows us to do the integral over k, so we get

$$
\begin{aligned}
S_{M1}^{(2)} = -ie^2 g_{\mu\nu} \int d^4x_1 \frac{e^{-ix_1\left(p_1-p_1'\right)}}{\left(p_1-p_1'\right)^2+i\varepsilon} \times \\
\frac{m}{V\sqrt{E_{\boldsymbol{p}_1'} E_{\boldsymbol{p}_2'}}} \bar{u}_{r_{1'},\alpha} \left(\boldsymbol{p}_1'\right) \gamma^{\mu}_{\alpha\beta} \bar{u}_{r_2',\delta} \left(\boldsymbol{p}_2'\right) e^{ip_2'x_1} \times \\
\frac{m}{V\sqrt{E_{\boldsymbol{p}_1} E_{\boldsymbol{p}_2}}} u_{r_1,\beta} \left(\boldsymbol{p}_1\right) u_{r_2,\epsilon} \left(\boldsymbol{p}_2\right) e^{-ip_2x_1} \gamma^{\nu}_{\delta\epsilon}
\end{aligned} \tag{17}
$$

We can now do the integral over x_1 to get another delta function:

$$
\int d^4x_1 e^{-ix_1\left(p_1-p_1'\right)} e^{ip_2'x_1} e^{-ip_2x_1} = (2\pi)^4 \delta^{(4)} \left(p_1'+p_2'-p_1-p_2\right) \tag{18}
$$

This expresses conservation of total momentum between the incoming and outgoing particles.

Finally, we can examine the matrix indices in 17. The combinations can be condensed into matrix notation by noting that

$$
\bar{u}_{r_{1'},\alpha} \left(\boldsymbol{p}_1'\right) \gamma^{\mu}_{\alpha\beta} u_{r_1,\beta} \left(\boldsymbol{p}_1\right) = \bar{u}_{r_{1'}} \left(\boldsymbol{p}_1'\right) \gamma^{\mu} u_{r_1} \left(\boldsymbol{p}_1\right) \tag{19}
$$

$$
\bar{u}_{r_2',\delta} \left(\boldsymbol{p}_2'\right) \gamma^{\nu}_{\delta\epsilon} u_{r_2,\epsilon} \left(\boldsymbol{p}_2\right) = \bar{u}_{r_2'} \left(\boldsymbol{p}_2'\right) \gamma^{\nu} u_{r_2} \left(\boldsymbol{p}_2\right) \tag{20}
$$

Thus the final form is

$$S_{M1}^{(2)} = -ie^2 g_{\mu\nu} (2\pi)^4 \delta^{(4)} \left(p_1' + p_2' - p_1 - p_2\right) \prod_{\text{electrons}} \sqrt{\frac{m}{VE}} \times$$
$$\frac{1}{\left(p_1 - p_1'\right)^2 + i\varepsilon} \bar{u}_{r_{1'}} \left(\boldsymbol{p}_1'\right) \gamma^\mu u_{r_1} \left(\boldsymbol{p}_1\right) \bar{u}_{r_2'} \left(\boldsymbol{p}_2'\right) \gamma^\nu u_{r_2} \left(\boldsymbol{p}_2\right) \tag{21}$$

The calculation for the RH side of Fig. 1 is (almost) obtained by observing that the diagram swaps p_1' and p_2'. The "almost" is because we need to swap $\bar{\psi}_{1'}^-$ and $\bar{\psi}_{2'}^-$ in 3, so that the term becomes

$$S_{M1}^{(2)} = -\frac{e^2}{2} \int d^4x_1 d^4x_2 \; N\left\{ (\bar{\psi}_{2'}^- \not{A} \psi_1^+)_{x_2} (\bar{\psi}_{1'}^- \not{A} \psi_2^+)_{x_1} \right\} \tag{22}$$

Because we've swapped two fermion fields, this introduces a minus sign into the overall result. Thus the term for the right-hand diagram is $S_{M1}^{(2)}$ with p_1' and p_2' swapped, multiplied by -1.

$$S_{M2}^{(2)} = ie^2 g_{\mu\nu} (2\pi)^4 \delta^{(4)} \left(p_1' + p_2' - p_1 - p_2\right) \prod_{\text{electrons}} \sqrt{\frac{m}{VE}} \times =$$
$$\frac{1}{\left(p_1 - p_2'\right)^2 + i\varepsilon} \bar{u}_{r_{2'}} \left(\boldsymbol{p}_2'\right) \gamma^\mu u_{r_1} \left(\boldsymbol{p}_1\right) \bar{u}_{r_1'} \left(\boldsymbol{p}_1'\right) \gamma^\nu u_{r_2} \left(\boldsymbol{p}_2\right) \tag{23}$$

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