EULER-LAGRANGE EQUATIONS FOR A FREE PARTICLE IN AN ELECTROMAGNETIC FIELD

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The treatment of a free particle in an electromagnetic field in the relativistic case closely follows that for the non-relativistic case. The difference is in the term representing the kinetic energy, which in the relativistic case includes the rest mass, so we have

$$L=-\frac{mc^{2}}{\gamma}+q\mathbf{A}\left(x\right)\cdot\mathbf{v}-qV\left(x\right) \tag{1}$$

where $\mathbf{A}\left(x\right)$ is the magnetic vector potential and $V\left(x\right)$ is the electric potential, both assumed to be functions of spacetime x. The particle's velocity is

$$\mathbf{v}=\dot{\mathbf{x}} \tag{2}$$

and γ is the usual relativistic factor

$$\gamma=\frac{1}{\sqrt{1-v^{2}/c^{2}}} \tag{3}$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial x_{i}}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_{i}}\right) \tag{4}$$

where the index i refers to the spatial coordinates.

From 1 we have

$$\frac{\partial L}{\partial x_{i}}=q\frac{\partial\left(\mathbf{A}\cdot\mathbf{v}\right)}{\partial x_{i}}-q\frac{\partial V}{\partial x_{i}} \tag{5}$$

We can write this as a vector equation by replacing $\partial/\partial x_{i}$ by ∇ and using the identity

$$\nabla\left(\mathbf{A}\cdot\mathbf{v}\right)=\left(\mathbf{A}\cdot\nabla\right)\mathbf{v}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{A}+\mathbf{A}\times\left(\nabla\times\mathbf{v}\right)+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) \tag{6}$$

Since we are taking $\mathbf{x}$ and $\mathbf{v}$ to be independent variables, any spatial derivatives of $\mathbf{v}$ are zero, so this reduces to

$$\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A}+\mathbf{v}\times(\nabla\times\mathbf{A}) \tag{7}$$
$$= (\mathbf{v}\cdot\nabla)\mathbf{A}+\mathbf{v}\times\mathbf{B} \tag{8}$$

where we used the expression for the magnetic field

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{9}$$

We therefore have

$$\nabla L = q(\mathbf{v}\cdot\nabla)\mathbf{A}+q\mathbf{v}\times\mathbf{B}-q\nabla V \tag{10}$$

The RHS of 4 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_i}\right) = \frac{d}{dt}\left(-\frac{\partial}{\partial v_i}\left(mc^2\sqrt{1-v^2/c^2}\right)+qA_i(x)\right) \tag{11}$$

We have

$$-\frac{\partial}{\partial v_i}\left(mc^2\sqrt{1-v^2/c^2}\right) = \frac{mv_i}{\sqrt{1-v^2/c^2}} = \gamma m v_i \tag{12}$$

We must take the total time derivative of the last term in 11 to get

$$\frac{d}{dt}(qA_i(x)) = q\left(\frac{\partial A_i}{\partial t}+\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}\right) \tag{13}$$
$$= q\left(\frac{\partial A_i}{\partial t}+\frac{\partial A_i}{\partial x_j}v_j\right) \tag{14}$$

where there is an implied sum over $j = 1,2,3$. In vector notation, we have

$$\frac{d}{dt}(q\mathbf{A}) = q\frac{\partial\mathbf{A}}{\partial t}+q(\mathbf{v}\cdot\nabla)\mathbf{A} \tag{15}$$

Inserting all this into 11 we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\mathbf{v}}\right) = \frac{d}{dt}(\gamma m\mathbf{v})+q\frac{\partial\mathbf{A}}{\partial t}+q(\mathbf{v}\cdot\nabla)\mathbf{A} \tag{16}$$

Equating 10 and 16 we get, after cancelling the $q(\mathbf{v}\cdot\nabla)\mathbf{A}$ term from both sides:

$$\frac{d}{dt}(\gamma m\mathbf{v})+q\frac{\partial\mathbf{A}}{\partial t} = q\mathbf{v}\times\mathbf{B}-q\nabla V \tag{17}$$

The electric field is given by

$$\mathbf{E} = -\nabla V-\frac{\partial\mathbf{A}}{\partial t} \tag{18}$$

so we end up with

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q\mathbf{v}\times\mathbf{B}+q\mathbf{E} \tag{19}$$

which is the Lorentz force law.

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