EXACT SOLUTION OF A ONE-FIELD MODEL

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As a first example of the theorem relating the evolution operator to connected Wick diagrams, which is

$$U_I(\infty,-\infty)=\sum(\text{all Wick diagrams})=:\exp\left(\sum\text{connected Wick diagrams}\right): \tag{1}$$

Coleman considers the simple interaction hamiltonian density

$$\mathscr{H}_I=g\rho(x)\phi(x) \tag{2}$$

where g is a numerical parameter giving the strength of the interaction, $\rho(x)$ is a numerical function (not an operator) which goes to zero in all four dimensions of spacetime and $\phi(x)$ is the usual field operator. Thus $\rho(x)$ determines over what regions of spacetime the interaction occurs, and the relative strength of the interaction at each of these points. g could have been absorbed into ρ, but Coleman chooses it to be separate.

In particular, as $\rho(x)$ depends on time as well as space, we can use it to implement the slow switching on and off of the interaction, as was done in the initial description of the S-matrix.

The useful feature of this system is that there are only 2 connected diagrams, no matter what order in perturbation theory we consider. The simplest diagram is a single vertex with a single edge and the other is two vertices connected by a contraction, as in Fig. 1. As there is only one field operator ϕ in the interaction term, any node in a Wick diagram can have only one edge connected to it.

The symmetry numbers are $S(D_1)=1$ (since only one diagram is possible with a single vertex and a single edge) and $S(D_2)=2$ (since interchanging the vertices 1 and 2 produces the same diagram). All higher order diagrams can be split into collections of these two connected diagrams.

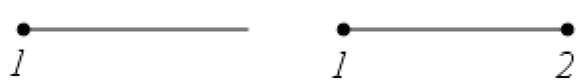

FIGURE 1. The two connected diagrams for Model 1.

Diagram 2 has the corresponding integral, which we'll call O_2:

$$O_2 = (-ig)^2 \int d^4x_1 d^4x_2 \overbrace{\phi(x_1)\,\phi}(x_2)\,\rho(x_1)\,\rho(x_2) \tag{3}$$

The contraction is just a number, so O_2 is the integral of purely numerical functions (no operators) and thus is itself some complex number, which Coleman writes as

$$O_2 = -\alpha + i\beta \tag{4}$$

The operator O_1 corresponding to diagram D_1, however, does contain an uncontracted operator. Coleman shows in eqns 8.63 to 8.67 that this comes out to

$$O_1 = \int d^3\boldsymbol{p}\left[-h(\boldsymbol{p})^* a_{\boldsymbol{p}} + h(\boldsymbol{p})\,a_{\boldsymbol{p}}^\dagger\right] \tag{5}$$

where $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^\dagger$ are the usual annihilation and creation operators for the scalar meson field, and

$$h(\boldsymbol{p}) = \frac{-ig\tilde{\rho}(\boldsymbol{p},\omega_{\boldsymbol{p}})}{(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}} \tag{6}$$

with $\tilde{\rho}(p)$ being the Fourier transform of $\rho(x)$. Since we have only one meson in diagram D_1, the four-momentum p must be 'on the mass shell', that is, it must satisfy

$$p^2 = \mu^2 \tag{7}$$

$$\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + \mu^2} \tag{8}$$

If the Fourier transform $\tilde{\rho}$ vanishes on the mass shell in 6, then $h = 0$ and therefore $O_1 = 0$. This is conservation of energy in this single-meson model. Because there is only one connected diagram in which any mesons are produced, and that diagram D_1 produces only a single meson, the only processes in which mesons are produced produces them one at a time. If $\mathscr{H}_i$ contained higher-order terms such as ϕ^2, then we could have two meson lines emanating from a single vertex and thus could produce 2 mesons in a single interaction.

There is a somewhat cryptic footnote in Coleman's book here, attempting to explain why a term such as $\rho\phi^2$ in the Hamiltonian would give rise to interactions where $\tilde{\rho}$ need not vanish on the mass shell in order for something to happen. I think what he's saying is that with a ϕ^2 term we can produce two mesons in a single interaction, since the lowest order of such a term contains a double creation operator $a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}'}^\dagger$. If we start with the vacuum state $|0\rangle$, then the energy for creating these two mesons comes from source term, which in momentum space is the Fourier transform $\tilde{\rho}$. The two mesons have four-momenta p_1 and p_2, and in each case we have

$$p_i^0 = \omega_{\boldsymbol{p}_i} = \sqrt{|\boldsymbol{p}_i|^2 + m^2} \tag{9}$$

Thus for two mesons, the total four-momentum is

$$p_T = p_1 + p_2 \tag{10}$$

Its components are

$$p_T^0 = p_1^0 + p_2^0 \tag{11}$$

$$= \sqrt{|\boldsymbol{p}_1|^2 + m^2} + \sqrt{|\boldsymbol{p}_2|^2 + m^2} \tag{12}$$

$$\boldsymbol{p}_T = \boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{13}$$

In this case, the mass-shell constraint is no longer valid, since

$$p_T^0 \neq \sqrt{|\boldsymbol{p}_T|^2 + 4m^2} \tag{14}$$

as you can verify by direct substitution. We have

$$\left(p_T^0\right)^2 = |\boldsymbol{p}_1|^2 + |\boldsymbol{p}_2|^2 + 2m^2 + 2\sqrt{|\boldsymbol{p}_1|^2 + m^2}\sqrt{|\boldsymbol{p}_2|^2 + m^2} \tag{15}$$

which certainly does not match the RHS of 14, at least unless $\boldsymbol{p}_1 = \boldsymbol{p}_2 = 0$.

Coleman now sets about evaluating O_1. The discussion is fairly easy to follow, but there are a few points that need some clarification.

Coleman considers the system starting from the ground state (vacuum state) $|0\rangle$ and shows in eqn 8.68 that the state $|\psi\rangle$ obtained by operating on this ground state with $S = U_I(\infty, -\infty)$ is

$$|\psi\rangle = A \exp\left[\int d^3\boldsymbol{p}\, h(\boldsymbol{p})\, a_{\boldsymbol{p}}^\dagger\right]|0\rangle \tag{16}$$

where A is a normalization constant. The derivation uses the fact that the annihilation operators $a_{\boldsymbol{p}}$ in ϕ all give zero when applied to the vacuum state.

We now write $|\psi\rangle$ as an integral over the basis states $|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\rangle$ where

$$|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\rangle = a_{\boldsymbol{p}_1}^\dagger a_{\boldsymbol{p}_2}^\dagger \ldots a_{\boldsymbol{p}_n}^\dagger |0\rangle \tag{17}$$

A general state $|\psi\rangle$ can consist of any number of mesons even though we can produce only a single meson at a time because $|\psi\rangle$ is the result of applying *all* Wick diagrams (including the disconnected ones) to the ground state.

The expansion in terms of this basis is eqn 8.70

$$|\psi\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3\boldsymbol{p}_1d^3\boldsymbol{p}_2\ldots d^3\boldsymbol{p}_n\psi^{(n)}\left(\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n\right)|\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n\rangle \tag{18}$$

where

$$\psi^{(n)}\left(\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n\right)\equiv\langle\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n|\psi\rangle=\langle\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n|S|0\rangle \tag{19}$$

In the arguments that follow, it's important to keep track of the presence or absence of the $n!$ factor, which can be a bit confusing. In 18, the multiple integral over the n different momenta would overcount the contribution from the nth order term, since if we interchange any pair of momenta, we get the same result. Since there are $n!$ ways of ordering the n momenta, we include the $\frac{1}{n!}$ to ensure that we get only one contribution from the nth order term.

Coleman now compares 16 with 18 to find the relation between $\psi^{(n)}$ and h. We get from 16

$$|\psi\rangle=A\sum_{n=0}^{\infty}\frac{1}{n!}\left[\int d^3\boldsymbol{p}h\left(\boldsymbol{p}\right)a_{\boldsymbol{p}}^{\dagger}\right]^n \tag{20}$$

Doing the comparison gives

$$\psi^{(0)}=A \tag{21}$$

$$\psi^{(1)}=Ah\left(\boldsymbol{p}\right) \tag{22}$$

$$\psi^{(2)}=Ah\left(\boldsymbol{p}_1\right)h\left(\boldsymbol{p}_2\right) \tag{23}$$

How do we get the $\psi^{(2)}$ equation? From 16, the second order term is

$$A\frac{1}{2!}\int d^3\boldsymbol{p}_1d^3\boldsymbol{p}_2h\left(\boldsymbol{p}_1\right)h\left(\boldsymbol{p}_1\right)a_{\boldsymbol{p}_1}^{\dagger}a_{\boldsymbol{p}_2}^{\dagger}|0\rangle=A\frac{1}{2!}\int d^3\boldsymbol{p}_1d^3\boldsymbol{p}_2h\left(\boldsymbol{p}_1\right)h\left(\boldsymbol{p}_1\right)|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle \tag{24}$$

Comparing this with 18 we see that the 2! has been separated out in both expressions, so we get 23.

The probability $P(n)$ of finding n mesons in the final state is the square modulus of the nth order term in 18. Coleman gives this as

$$P(n)=\frac{1}{n!}\int d^{3}\boldsymbol{p}_{1}d^{3}\boldsymbol{p}_{2}\ldots d^{3}\boldsymbol{p}_{n}\left|\psi^{(n)}\left(\boldsymbol{p}_{1},\boldsymbol{p}_{2},\ldots,\boldsymbol{p}_{n}\right)\right|^{2} \tag{25}$$

Again, if we left out the $\frac{1}{n!}$, we would be overcounting the number of final states, since interchanging any two momenta gives the same result after integration.

Coleman concludes by working out a couple of parameters. The normalization constant A comes out to

$$|A|^{2}=\exp\left(-\int d^{3}\boldsymbol{p}\left|h(\boldsymbol{p})\right|^{2}\right) \tag{26}$$

which makes α from 4 come out to

$$\alpha=\int d^{3}\boldsymbol{p}\left|h(\boldsymbol{p})\right|^{2} \tag{27}$$

and thus the probability $P(n)$ comes out to be a Poisson distribution

$$P(n)=e^{-\alpha}\frac{\alpha^{n}}{n!} \tag{28}$$

[Details in eqns 8.74 to 8.78.]

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