EXPECTATION VALUES IN THE INTERACTION PICTURE

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In the Schrödinger picture (SP), the equation of motion of the expectation value of an operator $\mathcal{O}^S$ is

$$\frac{d\bar{\mathcal{O}}}{dt} = {}_S\left\langle \Psi \middle| -i\left[\mathcal{O}^S, H^S\right] + \frac{\partial \mathcal{O}^S}{\partial t} \middle| \Psi \right\rangle_S \tag{1}$$

where everything is expressed in terms of the SP.

In the interaction picture (IP), the states and operators are given by

$$|\Psi\rangle_I = U_0^\dagger |\Psi\rangle_S \tag{2}$$

$$\mathcal{O}^I = U_0^\dagger \mathcal{O}^S U_0 \tag{3}$$

where

$$U_0 = e^{-iH_0 t} \tag{4}$$

and H_0 is the free Hamiltonian which is the same in both the SP and IP.

We can transform the RHS of 1 by using the above relations. We have

$$\frac{d\bar{\mathcal{O}}}{dt} = {}_I\left\langle \Psi \middle| U_0^\dagger \left(-i\left[U_0 \mathcal{O}^I U_0^\dagger, U_0 H^I U_0^\dagger\right] + \frac{\partial \mathcal{O}^S}{\partial t}\right) U_0 \middle| \Psi \right\rangle_I \tag{5}$$

The commutator term is

$$\begin{aligned} -iU_0^\dagger \left[U_0 \mathcal{O}^I U_0^\dagger, U_0 H^I U_0^\dagger\right] U_0 &= -iU_0^\dagger U_0 \mathcal{O}^I U_0^\dagger U_0 H^I U_0^\dagger U_0 + \\ &\quad iU_0^\dagger U_0 H^I U_0^\dagger U_0 \mathcal{O}^I U_0^\dagger U_0 && (6) \\ &= -i\mathcal{O}^I H^I + iH^I \mathcal{O}^I && (7) \\ &= -i\left[\mathcal{O}^I, H^I\right] && (8) \end{aligned}$$

This uses the fact that U_0 is unitary, so that $U_0^\dagger U_0 = I$, the identity operator.

Thus we have

$$\frac{d\bar{\mathcal{O}}}{dt} = {}_I\left\langle \Psi \middle| -i\left[\mathcal{O}^I, H^I\right] + U_0^\dagger \frac{\partial \mathcal{O}^S}{\partial t} U_0 \middle| \Psi \right\rangle_I \tag{9}$$

The second term is defined (see Klauber's equation 7-36) to be

$$\frac{\partial \mathcal{O}^I}{\partial t} \equiv U_0^\dagger \frac{\partial \mathcal{O}^S}{\partial t} U_0 \tag{10}$$

so we get the equation of motion for expectation values in the IP:

$$\frac{d\bar{\mathcal{O}}}{dt} = {}_I\left\langle \Psi \middle| -i\left[\mathcal{O}^I, H^I\right] + \frac{\partial \mathcal{O}^I}{\partial t} \middle| \Psi \right\rangle_I \tag{11}$$

In the SP, most operators have no explicit dependence on time, so $\frac{\partial \mathcal{O}^S}{\partial t} = 0$ usually. Thus the most common form of the equation of motion is

$$\frac{d\bar{\mathcal{O}}}{dt} = {}_I\left\langle \Psi \middle| -i\left[\mathcal{O}^I, H^I\right] \middle| \Psi \right\rangle_I \tag{12}$$

The expectation values themselves have the same form in both the SP and IP. In the SP, an expectation value is given by

$$\bar{\mathcal{O}} = {}_S\left\langle \Psi \middle| \mathcal{O}^S \middle| \Psi \right\rangle_S \tag{13}$$

Transforming to the IP, we have

$$\bar{\mathcal{O}} = {}_I\left\langle \Psi \middle| U_0^\dagger U_0 \mathcal{O}^I U_0^\dagger U_0 \middle| \Psi \right\rangle_I \tag{14}$$

$$= {}_I\left\langle \Psi \middle| \mathcal{O}^I \middle| \Psi \right\rangle_I \tag{15}$$