

FEYNMAN PROPAGATOR AS A CONTOUR INTEGRAL

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We've seen that we can write the Feynman propagator, defined as

$$i\Delta_F(x-y) \equiv \langle 0 | T [\phi(x) \phi^\dagger(y)] | 0 \rangle \quad (1)$$

in terms of commutators and that these commutators can be written as integrals over \mathbf{k} space as

$$i\Delta^\pm(x-y) = \frac{1}{2(2\pi)^3} \int d^3\mathbf{k} \frac{e^{\mp ik(x-y)}}{\omega_{\mathbf{k}}} \quad (2)$$

We can convert these integrals into contour integrals, which turn out to simplify things in the long run. The conversion is a bit tricky, so it bears close scrutiny.

First, we introduce a complex variable k_0 , which is the complex generalization of the energy component of the four-vector k . That is, the energy $\omega_{\mathbf{k}}$ is a point on the positive real axis in the complex plane where k_0 is the complex variable. (If you're used to doing complex variable stuff, the complex variable is usually denoted with the symbol z , so k_0 serves the role of z in what follows.) Now suppose we have some function $f(k_0)$ defined in this complex plane, and consider the contour integral

$$\int_{C^+} \frac{f(k_0)}{k_0 - \omega_{\mathbf{k}}} dk_0 \quad (3)$$

where C^+ is any counterclockwise contour that encloses the point $k_0 = \omega_{\mathbf{k}}$. From Cauchy's residue theorem, the residue of the integrand is

$$\text{Res} \frac{f(k_0)}{k_0 - \omega_{\mathbf{k}}} = \lim_{k_0 \rightarrow \omega_{\mathbf{k}}} (k_0 - \omega_{\mathbf{k}}) \frac{f(k_0)}{k_0 - \omega_{\mathbf{k}}} \quad (4)$$

$$= f(\omega_{\mathbf{k}}) \quad (5)$$

so the contour integral comes out to

$$\int_{C^+} \frac{f(k_0)}{k_0 - \omega_{\mathbf{k}}} dk_0 = 2\pi i f(\omega_{\mathbf{k}}) \quad (6)$$

[This assumes that $f(k_0)$ has no poles within the contour.]

Returning to 2, we'll consider Δ^+ for now. The exponent inside the integrand is the scalar product of two four-vectors, so written out in full it is

$$-ik(x-y) = -i\omega_{\mathbf{k}}(t_x - t_y) + i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}) \quad (7)$$

Therefore, we can write the integral as

$$i\Delta^+(x-y) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \left[\frac{e^{-i\omega_{\mathbf{k}}(t_x - t_y)}}{2\omega_{\mathbf{k}}} \right] d^3\mathbf{k} \quad (8)$$

If we define a function as follows:

$$f(k_0) \equiv \frac{e^{-ik_0(t_x - t_y)}}{k_0 + \omega_{\mathbf{k}}} \quad (9)$$

then the bracketed term in 8 is

$$f(\omega_{\mathbf{k}}) = \frac{e^{-i\omega_{\mathbf{k}}(t_x - t_y)}}{2\omega_{\mathbf{k}}} \quad (10)$$

We can now use 6 to write $f(\omega_{\mathbf{k}})$ as a contour integral of the function $f(k_0)$, that is

$$f(\omega_{\mathbf{k}}) = \frac{1}{2\pi i} \int_{C^+} \frac{f(k_0)}{k_0 - \omega_{\mathbf{k}}} dk_0 \quad (11)$$

$$i\Delta^+(x-y) = \frac{1}{2\pi i} \frac{1}{(2\pi)^3} \int e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \left[\int_{C^+} \frac{e^{-ik_0(t_x - t_y)}}{(k_0 + \omega_{\mathbf{k}})(k_0 - \omega_{\mathbf{k}})} dk_0 \right] d^3\mathbf{k} \quad (12)$$

At this point, Klauber writes the last line in the condensed form

$$i\Delta^+(x-y) = \frac{-i}{(2\pi)^4} \int_{C^+} \frac{e^{-ik(x-y)}}{k_0^2 - \omega_{\mathbf{k}}^2} d^4k \quad (13)$$

This form can be confusing, since it consists of a contour integral around C^+ in the (complex) k_0 plane, and a spatial integral over all of the (real) 3-d space of \mathbf{k} . It's important to remember that the integral over k_0 is *only* around the contour C^+ and *not* over the entire k_0 plane, while the integrals over the components of \mathbf{k} are over all *real* values of these components.

The denominator of 13 can now be transformed as follows, though I'm a bit unclear on why we can be sure this is allowed. First, we can write the four-vector scalar square modulus k^2 as

$$k^2 = k_0^2 - \mathbf{k}^2 \quad (14)$$

The reason this is a bit unclear is that k_0 , as used here, is a *complex* variable while \mathbf{k} is a real 3-vector. In any event, we can also use the relativistic energy-momentum relation

$$\omega_{\mathbf{k}}^2 - \mathbf{k}^2 = \mu^2 \quad (15)$$

and combine these two to get

$$k_0^2 - \omega_{\mathbf{k}}^2 = k^2 - \mu^2 \quad (16)$$

with the understanding that the k^2 on the RHS is a *complex* variable because it contains k_0^2 . This allows us to rewrite 13 as

$$i\Delta^+(x-y) = \frac{-i}{(2\pi)^4} \int_{C^+} \frac{e^{-ik(x-y)}}{k^2 - \mu^2} d^4k \quad (17)$$

The derivation for Δ^- is similar and given by Klauber in his section 3.17 (Appendix C to chapter 3). The result is

$$i\Delta^-(x-y) = \frac{i}{(2\pi)^4} \int_{C^-} \frac{e^{-ik(x-y)}}{k^2 - \mu^2} d^4k \quad (18)$$

where the contour C^- is a counterclockwise contour around the point $k_0 = -\omega_{\mathbf{k}}$.