FEYNMAN PROPAGATOR FOR FERMIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 6 January 2024.

The Feynman propagator for a virtual fermion (electron or positron) is derived in Klauber's Chapter 4, and is, in momentum space

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} \tag{1}$$

Here, p is the 4-momentum of the fermion and m is its mass. The ε is an infinitesimal quantity. The slash notation is shorthand for

$$\not{p} \equiv p_\mu \gamma^\mu \tag{2}$$

where γ^μ are the 4×4 gamma matrices. Thus $S_F(p)$ in 1 is actually a 4×4 matrix equation, so to be completely accurate, we should write it as

$$S_F(p) = \frac{\not{p} + mI}{p^2 - m^2 + i\varepsilon}$$

where I is the 4×4 identity matrix.

We can write this in an equivalent form as follows. First we work out

$$\not{p}^2 = p_\mu \gamma^\mu p_\nu \gamma^\nu \tag{3}$$
$$= \gamma^\mu \gamma^\nu p_\mu p_\nu \tag{4}$$
$$= \frac{1}{2}\left(\gamma^\mu \gamma^\nu p_\mu p_\nu + \gamma^\nu \gamma^\mu p_\nu p_\mu\right) \tag{5}$$
$$= \frac{1}{2}\left(\gamma^\mu \gamma^\nu p_\mu p_\nu + \gamma^\nu \gamma^\mu p_\mu p_\nu\right) \tag{6}$$
$$= \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} p_\mu p_\nu \tag{7}$$

where $\{\gamma^\mu, \gamma^\nu\}$ is the anticommutator of the two gamma matrices. In this derivation, note that p_μ is just a number so it commutes with everything, while the γ^μ are 4×4 matrices.

We've seen earlier that

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu} I \tag{8}$$

where $g^{\mu\nu}$ is the usual metric tensor from special relativity, and I is the 4×4 identity matrix. From 7 we have

$$\not{p}^2 = \frac{1}{2}\left(2g^{\mu\nu}I\right)p_\mu p_\nu \tag{9}$$

$$= g^{\mu\nu}p_\mu p_\nu I \tag{10}$$

$$= p^2 I \tag{11}$$

We can therefore write 1 as

$$S_F(p) = \left(\not{p}+mI\right)\left[\left(p^2-m^2+i\varepsilon\right)I\right]^{-1} \tag{12}$$

$$= \left(\not{p}+mI\right)\left(\not{p}^2-m^2I+i\varepsilon I\right)^{-1} \tag{13}$$

The second factor contains a difference of squares, but it is the difference of squares of matrix objects, so we should check that the usual algebraic relation for factoring a difference of squares applies also to a matrix relation. We have

$$\left(\not{p}-mI\right)\left(\not{p}+mI\right) = \left(p_\mu\gamma^\mu - mI\right)\left(p_\nu\gamma^\nu + mI\right) \tag{14}$$

$$= p_\mu\gamma^\mu p_\nu\gamma^\nu + p_\mu\gamma^\mu mI - mIp_\nu\gamma^\nu - m^2I^2 \tag{15}$$

The identity matrix I commutes with everything, $I^2 = I$, and since μ and ν are dummy summation indices, the middle two terms cancel, and we're left with

$$\left(\not{p}-mI\right)\left(\not{p}+mI\right) = p_\mu\gamma^\mu p_\nu\gamma^\nu - m^2I \tag{16}$$

$$= \not{p}^2 - m^2 I \tag{17}$$

Thus the factoring relation works for this matrix equation as well. [It's worth noting that this does *not* work if the two matrices in each factor don't commute.]

Returning to 13, we can write the second factor as (disregarding the infinitesimal)

$$\left(\not{p}^2-m^2I\right)^{-1} = \left[\left(\not{p}-mI\right)\left(\not{p}+mI\right)\right]^{-1} \tag{18}$$

For any two square invertible matrices A and B, we have

$$(AB)^{-1} = B^{-1}A^{-1} \tag{19}$$

With

$$\begin{aligned} A &= (\not{p} - mI) \\ B &= (\not{p} + mI) \end{aligned} \tag{20}$$

we have

$$S_F(p) = (\not{p} + mI)(\not{p} + mI)^{-1}(\not{p} - mI)^{-1} \tag{21}$$

$$= (\not{p} - mI)^{-1} \tag{22}$$

Klauber writes this in Chapter 12 onwards (dropping the explicity I): as

$$S_F(p) = \frac{1}{\not{p} - m + i\varepsilon} \tag{23}$$

Klauber also replaces m (the experimentally measured mass) with m_0 (the 'bare' mass, which isn't experimentally measurable).

It must be remembered that this is actually a matrix equation, so the quantity in the denominator is actually a matrix inverse.