FEYNMAN RULES FOR LOOPS

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The Feynman rules in the earlier post applied to QED diagrams without loops. Here, we'll add in the extra rules needed to deal with loops of various types.

Fig. 1 shows an electron closed loop. It is an order-2 diagram in which an incoming electron emits a virtual electron and a virtual photon, which combine again to form the outgoing electron. The important point here is that the momentum k of the virtual photon (and hence also the virtual electron) is not fixed by the diagram; that is k can be anything, as long as the virtual electron's momentum is $p-k$, so that momentum is conserved at both vertices. As such, when we form the S matrix element, we need to integrate over k. This gives rise to the Feynman rule for internal particles:

For each 4-momentum q (or k in Fig. 1) that is not fixed by momentum conservation, do an integral of the form

$$\frac{1}{(2\pi)^4}\int d^4q \tag{1}$$

over this momentum.

Applying this additional rule to Fig. 1 gives us the amplitude

$$(2\pi)^4\delta^{(4)}\left(p'-p\right)\sqrt{\frac{m}{VE_{\boldsymbol{p}}}}\sqrt{\frac{m}{VE_{\boldsymbol{p}'}}}\mathcal{M} \tag{2}$$

where

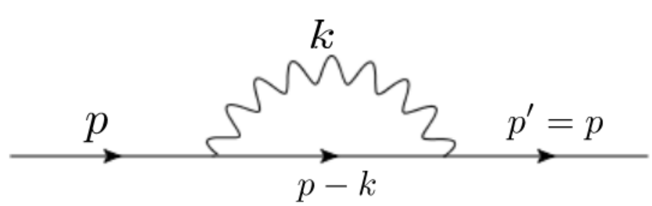

FIGURE 1. Electron closed loop.

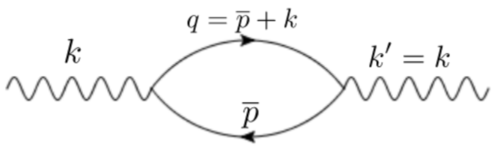

FIGURE 2. Photon closed loop.

$$\mathcal{M} = \bar{u}_r\left(\boldsymbol{p}'\right) ie\gamma^{\mu}\left[\frac{1}{(2\pi)^4}\int d^4k\; iD_{F\mu\nu}(k)\, iS_F(p-k)\right] ie\gamma^{\nu} u_r(\boldsymbol{p}) \tag{3}$$

Here, $D_{F\mu\nu}$ is the photon propagator, S_F is the fermion propagator and $\bar{u}_r$ and u_r are the electron spinors.

Another kind of closed loop is shown in Fig. 2. An incoming photon splits into a virtual electron-positron pair which recombine to produce an outgoing photon. Note that although momentum conservation appears to be violated if we read time going from left to right, it is conventional to regard positrons as moving backwards in time in these diagrams. Thus the momentum $\bar{p}$ of the positron is actually $-\bar{p}$ when viewed in our forward-time frame. It's easiest to think of these diagrams by following the direction of the arrows on fermion paths. Thus on the left hand vertex, there is a real photon k and a virtual positron $\bar{p}$ entering, and a virtual electron leaving. Thus momentum conservation requires the particle leaving to have the momentum of the two particles entering, so $q = \bar{p} + k$.

Similarly, at the right hand vertex, a virtual electron $q = \bar{p} + k$ enters and a virtual positron $\bar{p}$ and real photon k leaves.

Note that this convention applies *only* to virtual positrons; other types of virtual particles and all real particles (including positrons) have their momenta given in the forward time direction.

There is an additional Feynman rule for loops containing only virtual fermions (without any virtual photons). It is:

For each closed loop of internal fermions only, take the trace in spinor space of the matrix resulting from the product of the fermion propagators, and multiply by -1. Applying these rules to Fig. 2 we have the amplitude

$$(2\pi)^4\,\delta^{(4)}\left(k'-k\right)\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}}\frac{1}{\sqrt{2V\omega_{\boldsymbol{k}'}}}\mathcal{M} \tag{4}$$

where

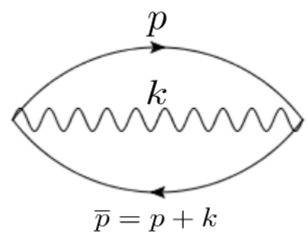

FIGURE 3. Vacuum bubble.

$$\mathcal{M} = \varepsilon_{\mu,r'}\left(\boldsymbol{k}'\right)\frac{-1}{(2\pi)^4}\text{Tr}\left[\int d^4\bar{p}\; iS_F\left(\bar{p}\right)ie\gamma^{\mu}iS_F\left(\bar{p}+k\right)ie\gamma^{\nu}\right]\varepsilon_{\nu,r}\left(\boldsymbol{k}\right) \tag{5}$$

$$= -\frac{e^2}{(2\pi)^4}\text{Tr}\left[\int d^4\bar{p}\; iS_F\left(\bar{p}\right)ie\gamma^{\mu}iS_F\left(\bar{p}+k\right)ie\gamma^{\nu}\right]\varepsilon_{\mu,r'}\left(\boldsymbol{k}'\right)\varepsilon_{\nu,r}\left(\boldsymbol{k}\right) \tag{6}$$

Recall that the μ or ν subscript on the polarization vectors refers to a component of that vector, so that $\mathcal{M}$ is actually a single number, rather than a vector or matrix. The trace must be evaluated separately for each combination of μ and ν, and this value is then multiplied by the corresponding components of the polarization vectors.

A final example is shown in Fig. 3. There are no external vertices, so this diagram represents a virtual electron, positron and photon emerging spontaneously from the vacuum and then all recombining to disappear back into the vacuum. When applying the Feynman rules to the vacuum bubble, we note that there are no external lines, so all three virtual particles are represented by propagators. The electron-positron loop is a fermion loop as above, so we invoke the trace rule here. Thus we have

$$(2\pi)^4\,\delta^{(4)}(0)\,\mathcal{M} \tag{7}$$

where

$$\mathcal{M} = \frac{-1}{(2\pi)^8}\int d^4k\int d^4p\;\text{Tr}\left[iS_F\left(p+k\right)ie\gamma^{\mu}iS_F\left(p\right)ie\gamma^{\nu}\right]iD_{F\mu\nu}\left(k\right) \tag{8}$$

$$= \frac{-e^2}{(2\pi)^8}\int d^4k\int d^4p\;\text{Tr}\left[S_F\left(p+k\right)\gamma^{\mu}S_F\left(p\right)\gamma^{\nu}\right]iD_{F\mu\nu}\left(k\right) \tag{9}$$

The factor of -1 comes from applying the trace rule above.

Thus the complete amplitude is

$$\frac{-e^2}{(2\pi)^4}\delta^{(4)}(0)\int d^4k\int d^4p\,\text{Tr}\left[S_F(p+k)\gamma^\mu S_F(p)\gamma^\nu\right]iD_{F\mu\nu}(k) \qquad (10)$$

As with the example above, the trace produces a single number for each combination of μ and ν, and this number is multiplied by the corresponding propagator $D_{F\mu\nu}$.