FIELD OPERATORS FOR THE INFINITE SQUARE WELL

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The creation and annihilation operators $\hat{a}_{\mathbf{p}}^{\dagger}$ and $\hat{a}_{\mathbf{p}}$ create and annihilate a particle in a specific momentum state so, because of the uncertainty principle, such states are completely unlocalized in position. We can construct analogous operators that create and annihilate a particle at a specific position. Such operators are called *field operators*. A field operator, operating at a precise position, must therefore be completely unlocalized in momentum.

The creation and annihilation field operators for a particle in a 3-d infinite square well are defined as

$$\hat{\psi}^{\dagger}(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{1}$$

$$\hat{\psi}(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2}$$

where $\mathcal{V}$ is the volume of the square well.

To see that they actually do create and annihilate a particle at position $\mathbf{x}$ we'll have a look at their effect when operating on particular states. First, we'll apply $\hat{\psi}^{\dagger}(x)$ to the vacuum state.

$$|\Psi\rangle \equiv \hat{\psi}^{\dagger}(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{a}_{\mathbf{p}}^{\dagger}|0\rangle \tag{3}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle \tag{4}$$

We can now insert the unit operator in the form

$$1 = \int d^3\mathbf{y}\,|\mathbf{y}\rangle\langle\mathbf{y}| \tag{5}$$

and we get

$$\hat{\psi}^{\dagger}(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathcal{V}}}\int d^3\mathbf{y}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{y}\rangle\langle\mathbf{y}|\mathbf{p}\rangle \tag{6}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\int d^3\mathbf{y}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{y}\rangle\left[\frac{1}{\sqrt{\mathcal{V}}}e^{i\mathbf{p}\cdot\mathbf{y}}\right] \tag{7}$$

$$= \int d^3\mathbf{y}\left[\frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\right]|\mathbf{y}\rangle \tag{8}$$

$$= \int d^3\mathbf{y}\,\delta^{(3)}(\mathbf{y}-\mathbf{x})|\mathbf{y}\rangle \tag{9}$$

$$= |\mathbf{x}\rangle \tag{10}$$

So the creation operator $\hat{\psi}^{\dagger}(\mathbf{x})$ operating on the vacuum state creates a single particle at position $\mathbf{x}$.

We can also see that 4 represents a single particle by applying the number operator $\hat{n}_{\boldsymbol{q}} = \hat{a}^{\dagger}_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}}$ to $|\Psi\rangle$ and then summing over all momenta $\boldsymbol{q}$. We have

$$\sum_{\boldsymbol{q}}\hat{n}_{\boldsymbol{q}}|\Psi\rangle = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\boldsymbol{q}}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}}\hat{a}^{\dagger}_{\mathbf{p}}|0\rangle \tag{11}$$

If we consider the two operators $\hat{a}_{\boldsymbol{q}}\hat{a}^{\dagger}_{\mathbf{p}}$, we can use their commutator

$$\left[\hat{a}_{\boldsymbol{q}},\hat{a}^{\dagger}_{\boldsymbol{p}}\right] = \delta_{\boldsymbol{qp}} \tag{12}$$

We then have

$$\hat{a}_{\boldsymbol{q}}\hat{a}^{\dagger}_{\mathbf{p}}|0\rangle = \left(\delta_{\boldsymbol{qp}} + \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\boldsymbol{q}}\right)|0\rangle \tag{13}$$

$$= \delta_{\boldsymbol{qp}}|0\rangle \tag{14}$$

because $\hat{a}_{\boldsymbol{q}}|0\rangle = 0$. Applying this to 11, we have

$$\sum_{\boldsymbol{q}}\hat{n}_{\boldsymbol{q}}|\Psi\rangle = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\boldsymbol{q}}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\boldsymbol{q}}\delta_{\boldsymbol{qp}}|0\rangle \tag{15}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\boldsymbol{p}}|0\rangle \tag{16}$$

$$= |\Psi\rangle \tag{17}$$

where the last line follows by comparison with 4. Thus $|\Psi\rangle$ is an eigenstate of $\sum_{\boldsymbol{q}}\hat{n}_{\boldsymbol{q}}$ with eigenvalue 1, so it does indeed represent a single particle.

We can now try the annihilation operator $\hat{\psi}(\mathbf{y})$ operating on a one-particle state $|\mathbf{x}\rangle$. We get

$$\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{x}\rangle \tag{18}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}.\mathbf{q}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{q}\rangle\langle\mathbf{q}|\mathbf{x}\rangle \tag{19}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}.\mathbf{q}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{q}\rangle\left[\frac{1}{\sqrt{\mathcal{V}}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right] \tag{20}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}}e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})}\hat{a}_{\mathbf{p}}|\mathbf{q}\rangle \tag{21}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p},\mathbf{q}}e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})}\delta_{\mathbf{pq}}|0\rangle \tag{22}$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}|0\rangle \tag{23}$$

$$= \delta^{(3)}(\mathbf{y}-\mathbf{x})|0\rangle \tag{24}$$

where in the fourth line the momentum annihilation operator $\hat{a}_{\mathbf{p}}$ operating on $|\mathbf{q}\rangle$ produces the vacuum state only if $\mathbf{p}=\mathbf{q}$; otherwise it produces zero.

Thus $\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle$ produces the vacuum state if $\mathbf{y}=\mathbf{x}$ and zero otherwise.