FINAL FORM FOR THE SCALAR QUANTUM FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 3 August 2021.

From the assumptions that the scalar quantum field must be a linear combination of creation and annihilation operators, and that it must transform properly under Lorentz transformations and translations, we arrived at a form for the scalar quantum field as follows:

$$\phi(x)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}\alpha(p)+ge^{ip\cdot x}\alpha^\dagger(p)\right] \tag{1}$$

We still need to satisfy what might be seen as the two most important requirements. First, the field must be hermitian, since it's an observable, and second, the field operators $\phi(x)$ and $\phi(y)$ must be independent of each other (that is, they must commute) if the spacetime interval $x-y$ is spacelike, so that $(x-y)^2<0$. That is

$$[\phi(x),\phi(y)]=0 \quad \text{if } (x-y)^2<0 \tag{2}$$

To satisfy these conditions, Coleman proposes two independent versions for ϕ in his equation 3.34. He reverts to the a instead of α for the operators, using

$$\begin{aligned}\alpha^\dagger(p)&\equiv(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}^\dagger\\ \alpha(p)&\equiv(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}\end{aligned} \tag{3}$$

These use the partial fields

$$\phi^{(+)}(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{4}$$

$$\phi^{(-)}(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} \tag{5}$$

Note that

$$\phi^{(+)}(x)^\dagger=\phi^{(-)}(x) \tag{6}$$

We then have

$$\phi(x)=f\phi^{(+)}(x)+g\phi^{(-)}(x) \tag{7}$$

In order to satisfy hermiticity, we need

$$\phi^{\dagger}(x)=\phi(x) \tag{8}$$

from which we see that

$$f=g^{\dagger} \tag{9}$$

Since any multiple of $\phi(x)$ is a valid field, we might as well take

$$|f|=|g|=1 \tag{10}$$

This means that the most general form for the field is

$$\phi(x)=e^{i\theta}\phi^{(+)}(x)+e^{-i\theta}\phi^{(-)}(x) \tag{11}$$

where θ is any real number.

Two independent forms for the field are ϕ^1 with $\theta=0$ and ϕ^2 with $\theta=\frac{\pi}{2}$:

$$\phi^1(x)=\phi^{(+)}(x)+\phi^{(-)}(x) \tag{12}$$

$$\phi^2(x)=i\left(\phi^{(+)}(x)-\phi^{(-)}(x)\right) \tag{13}$$

We now need to evaluate the commutators of these two possibilities to see if 2 is satisfied. We can do this using the commutators of the creation and annihilation operators:

$$\begin{aligned}\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right]&=\delta\left(\boldsymbol{p}-\boldsymbol{p}'\right)\\ \left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}\right]=\left[a_{\boldsymbol{p}}^{\dagger},a_{\boldsymbol{p}'}^{\dagger}\right]&=0\end{aligned} \tag{14}$$

The first possibility is that both ϕ^1 and ϕ^2 (and any linear combination of them) represents an observable. In order for that to be possible, all spacelike commutators between linear combinations of ϕ^1 and ϕ^2 must be zero. In particular, we must have

$$\left[\phi^1(x),\phi^2(y)\right]=0 \quad \text{if } (x-y)^2<0 \tag{15}$$

Coleman works out this commutator in detail in equations 3.37 and 3.38, and shows that, the commutator comes out to

$$\left[\phi^1(x),\phi^2(y)\right]=-i\Delta_+(x-y)-i\Delta_+(y-x) \tag{16}$$

where

$$\Delta_+(x-y)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}e^{-ip\cdot(x-y)} \tag{17}$$

The time derivative of Δ_+ comes out to

$$\frac{\partial}{\partial x^0}\Delta_+(x)=-\frac{i}{2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}e^{-ip\cdot x} \tag{18}$$

which is the same integral we looked at earlier using contour integration, and determined that it isn't zero. If a function has a non-zero derivative, then the function itself cannot be identically zero, so using the two independent forms 12 doesn't give us a field that satisfies causality.

The only correct solution is to use ϕ^1 on its own, and we find (Coleman equation 3.42) that (calling ϕ^1 just ϕ):

$$[\phi(x),\phi(y)]=\Delta_+(x-y)-\Delta_+(y-x)\equiv i\Delta(x-y) \tag{19}$$

This *does* turn out to be zero for spacelike intervals, according to the following argument. First, we observe that Δ_+, and therefore Δ, is Lorentz invariant, since from its definition 17, it's the integral of the Lorentz invariant measure $d^3\boldsymbol{p}/2\omega_{\boldsymbol{p}}$ multiplied by the exponential of a Lorentz scalar $p\cdot(x-y)$. A Lorentz invariant quantity can be worked out in any inertial frame. A spacelike interval is one where the two events must always be separated by a non-zero spatial distance, no matter what inertial frame you use. However, the two events can occur in either order (or simultaneously) with regard to time. As shown in footnote 8 in the book, any spacelike interval can be turned into its negative by a proper Lorentz transformation, so $x-y$ and $y-x$ are related by a single proper Lorentz transformation. Thus $\Delta_+(x-y)=\Delta_+(y-x)$ and thus $\Delta(x-y)=0$.

To see that this is possible, it might help to look at two events, where $e_1=(t,x,y,z)=(0,0,0,0)$ and $e_2=(0,1,0,0)$. In the frame in which these events have these coordinates, they are simulaneous but have a non-zero separation in space, so e_2-e_1 is a spacelike interval. From the argument above, it should be possible to find a Lorentz transformation that turns e_2-e_1 into e_1-e_2. The usual Lorentz transformation for a boost is given as

$$\begin{aligned}x'&=\gamma(x-vt)\\ t'&=\gamma(t-vx)\end{aligned} \tag{20}$$

Since $t=0$ for both events and $\gamma>0$ for any velocity v, it won't matter what the relative velocity of two frames is, we'll always have $e_1'=(0,0,0,0)$ and $e_2'=(-\gamma v,\gamma,0,0)$ so that $e_2'-e_1'$ will never be equal to e_1-e_2. Thus we can't invert a spacelike vector using a boost.

However, remember that an ordinary rotation in space is also a proper Lorentz transformation, and we can turn a vector pointing in the $+x$ direction into an equal and opposite vector in the $-x$ direction by rotating it by π about the z axis (or the y axis).

The same argument doesn't work for two spacetime points with a time-like separation, since there is always a non-zero time (of the same sign) between these two events. In other words, one event can be connected to the other by a light signal, so we can't reverse the temporal order of the events, meaning that we can't transform $x-y$ to $y-x$ by a single Lorentz transformation.

We therefore end up with the final form for our scalar quantum field:

$$\boxed{\phi(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right]} \tag{21}$$

This field is often written as the sum of two parts:

$$\phi(x)=\phi^{(+)}(x)+\phi^{(-)}(x) \tag{22}$$

where

$$\phi^{(+)}(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{23}$$

$$\phi^{(-)}(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x} \tag{24}$$

It seems counterintuitive to name the + field as the one with the annihilation operators, but the rationale appears to be that if we apply the Schrödinger energy operator to this field we get

$$i\frac{\partial}{\partial t}\phi^{(+)}(x)=i\frac{\partial}{\partial t}\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{25}$$

$$=\int\frac{i\,d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}\frac{\partial}{\partial t}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{26}$$

$$=\int\frac{(+\omega_{\boldsymbol{p}})\,d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{27}$$

That is, we get a $+\omega_{\boldsymbol{p}}$ after applying the operator $i\frac{\partial}{\partial t}$, so we get a positive frequency, hence the + sign. Applying the Schrödinger energy operator to $\phi^{(-)}$ brings in a factor of $-\omega_{\boldsymbol{p}}$, that is, a negative frequency.

PINGBACKS

Pingback: Free scalar field as the fundamental object
Pingback: Currents from spacetime translations and the energy-momentum tensor
Pingback: Continuous internal symmetries
Pingback: Normal ordering
Pingback: Mass renormalization
Pingback: Scalar fields - summary
Pingback: Wick diagrams to Feynman diagrams
Pingback: Evaluating Feynman diagrams