FREE PARTICLES AND INTERACTION TERMS IN LAGRANGIANS

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One feature of Lagrangians in both classical and quantum field theories that doesn't seem to be very well explained in any book I've looked at is the reason that Lagrangians representing free (that is, non-interacting) fields and interacting fields have the forms that they do. The best explanation I've found is in Chapter 7 of Lancaster & Blundell's book, although even here, it's mentioned only in passing.

To see what I mean, we can start with a Lagrangian (actually, the Lagrangian density, but we'll assume this in what follows) for a free, massless field:

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\partial_{\mu}\phi = \frac{1}{2}\left(\partial^{\mu}\phi\right)^{2} \tag{1}$$

Here ϕ is the field (classical in this case, although the same reasoning applies to quantum fields) and the Greek index μ ranges over the four coordinates of spacetime, as usual.

To get the equations of motion, we apply the Euler-Lagrange equation, which in the case of a single field is

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial_{\mu}\phi} = 0 \tag{2}$$

The components of this equation are

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} &= 0 \\ \frac{\partial\mathcal{L}}{\partial_{\mu}\phi} &= \partial^{\mu}\phi \end{aligned} \tag{3}$$

So the Euler-Lagrange equation gives us the equation of motion

$$\partial^{\mu}\partial_{\mu}\phi = 0 \tag{4}$$

In terms of spacetime coordinates, this is

$$\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=0 \tag{5}$$

This is the wave equation in 3 dimensions, with a wave speed of $v=1$, which is the speed of light if we use relativistic units.

It has solutions of the form

$$\phi(x,t)=\sum_{\boldsymbol{p}}a_{\boldsymbol{p}}e^{-i(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})} \tag{6}$$

where the $a_{\boldsymbol{p}}$ are constants. The quantities $E_{\boldsymbol{p}}$ and $\boldsymbol{p}$ are usually interpreted as the energy and momentum, but if all we're after is a solution to 5, they can be any constants, provided that

$$E_{\boldsymbol{p}}=|\boldsymbol{p}| \tag{7}$$

This is the usual dispersion relation for a photon.

The key feature of 5 is that it's *linear*, in the sense that the solution 6 is a linear combination of functions, each of which is itself a solution. Because the RHS of 5 is zero, and the field ϕ appears on the LHS only as a linear term (no terms like ϕ^2, $\sqrt{\phi}$, etc), we can add together individual solutions to get another solution. It also means that the various solutions obey the principle of superposition; introducing a second field merely adds its effects to the first, without any interaction between the two fields.

Returning to 2, we see that if a Lagrangian contains only terms that are quadratic in the field and linear in its second derivative, then the Euler-Lagrange equation will contain only terms linear in the field and its first derivatives. In other words, such a Lagrangian will always give rise to equations of motion in which the various solutions do not interact.

As a second example, we can introduce a mass into the Lagrangian above by writing

$$\mathcal{L}=\frac{1}{2}(\partial^\mu\phi)^2-\frac{1}{2}m^2\phi^2 \tag{8}$$

We now have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi}&=-m^2\phi\\ \frac{\partial\mathcal{L}}{\partial_\mu\phi}&=\partial^\mu\phi\end{aligned} \tag{9}$$

The equation of motion is

$$-m^2\phi-\partial^\mu\partial_\mu\phi=0 \tag{10}$$

or, cancelling the minus signs

$$m^2\phi+\partial^\mu\partial_\mu\phi=0 \tag{11}$$

Again, the field and its derivatives appear as linear terms, so again, this represents a system of non-interacting fields obeying superposition. The solutions are

$$\phi(x,t)=\sum_{\boldsymbol{p}}a_{\boldsymbol{p}}e^{-i(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})} \tag{12}$$

but this time, the dispersion relation is

$$E_{\boldsymbol{p}}=\sqrt{p^2+m^2} \tag{13}$$

Now suppose we modify the original massless Lagrangian by adding a term:

$$\mathcal{L}=\frac{1}{2}(\partial^\mu\phi)^2-\frac{1}{2}m^2\phi^2+J(x)\phi \tag{14}$$

We now have

$$-\frac{\partial\mathcal{L}}{\partial\phi}=m^2\phi(x)-J(x) \tag{15}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=\partial_\mu\partial^\mu\phi \tag{16}$$

The equation of motion is therefore

$$\left(\partial_\mu\partial^\mu\phi+m^2\right)\phi(x)=J(x) \tag{17}$$

Although the LHS is the same as 11, this equation no longer has simple solutions. If we found two solutions $\phi_1(x)$ and $\phi_2(x)$, for example, then their sum no longer satisfies 17; rather, we would have

$$\left(\partial_\mu\partial^\mu\phi+m^2\right)(\phi_1(x)+\phi_2(x))=2J(x) \tag{18}$$

Thus the introduction of a second field interacts with the first field, and fields no longer obey superposition.

The function $J(x)$ is known as a *source term* and doesn't involve any fields. However, we can also introduce terms that are non-quadratic powers of the field, such as ϕ^4. Or, we could introduce two different fields ϕ_1 and ϕ_2 and add a term to the Lagrangian linking them, as in

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\phi_{1}\right)^{2}-\frac{1}{2}m^{2}\phi_{1}^{2}+\frac{1}{2}\left(\partial_{\mu}\phi_{2}\right)^{2}-\frac{1}{2}m^{2}\phi_{2}^{2}-g\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2} \tag{19}$$

The first four terms are just the Lagrangian terms for non-interacting fields, but the last term $g\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2}$ contains terms that are not quadratic in the fields, and thus will produce extra terms in the equations of motion that prohibit linear solutions, indicating that the two fields interact.

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