FREE SCALAR FIELD AS THE FUNDAMENTAL OBJECT

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In Section 3.4, Coleman says he will rebuild the theory of the quantum scalar field by taking the field $\phi(x)$ as the fundamental object and rederiving the other properties from that. I have to confess I'm somewhat mystified by this section, as he appears to use the solution for the free field that was obtained earlier to derive itself.

He starts with the solution found earlier, namely

$$\phi(x)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right] \tag{1}$$

He then shows that this field satisfies the Klein-Gordon equation, which is

$$\Box^{2}\phi(x)+\mu^{2}\phi(x)=0 \tag{2}$$

where $\Box^2$ is the d'Alembertian operator:

$$\Box^{2}\equiv\partial^{\mu}\partial_{\mu} \tag{3}$$

He does this by rewriting 1 in its original 4-dim form as

$$\phi(x)=\int\frac{d^{4}p}{(2\pi)^{3}}\delta\left(p^{2}-\mu^{2}\right)\theta\left(p^{0}\right)\left[\alpha(p)e^{-ip\cdot x}+\alpha^{\dagger}(p)e^{ip\cdot x}\right] \tag{4}$$

and substituting this into 2. This is shown in eqns 3.49 and 3.50.

Next, he takes the commutator that we worked out earlier

$$[\phi(x),\phi(y)]=i\Delta(x-y) \tag{5}$$

$$=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right] \tag{6}$$

which we have already shown to be zero if $(x-y)$ is a spacelike interval, so that $(x-y)^2<0$.

Coleman takes 2 and 6 as postulates, along with the requirement that ϕ be hermitian, and that it transforms properly under translations and Lorentz transformations. He then sketches out an argument (from eqns 3.52 to 3.56), saying that these requirements allow us to derive the commutation relations for the operators

$$\begin{aligned}\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right] &= \delta\left(\boldsymbol{p}-\boldsymbol{p}'\right)\\ \left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}\right] &= \left[a_{\boldsymbol{p}}^{\dagger},a_{\boldsymbol{p}'}^{\dagger}\right]=0\end{aligned} \tag{7}$$

The argument appears to be valid, but I can't really see the point. We already have the solution 1 for the field, so I don't see that anything has been gained by going through the argument in reverse.

He goes on a bit further by saying that the commutation condition 6 can be weakened to a pair of *equal-time* commutators. That is, instead of $[\phi(x),\phi(y)]$ we can calculate $[\phi(\boldsymbol{x},t),\phi(\boldsymbol{y},t)]$ where the time component of both fields in the commutator is the same. We can do this because, for any spacelike interval, we can always find a Lorentz transformation that converts the interval to one where both events that define the interval are simultaneous.

Since 2 is a second-order differential equation, a complete solution is found when we have the general form of the solution, and also initial conditions for the field and for the field's time derivative. Therefore, if we can find

$$[\phi(\boldsymbol{x},t),\phi(\boldsymbol{y},t)] = i\Delta(\boldsymbol{x}-y) \tag{8}$$

$$\left[\dot{\phi}(\boldsymbol{x},t),\phi(\boldsymbol{y},t)\right] = i\,\frac{\partial}{\partial x^0}\Delta(x-y)\bigg|_{x^0=y^0} \tag{9}$$

we will have a complete solution for $\phi(x)$.

He calculates these commutators in eqns 3.60 and 3.61, although these equations contain an error. If we start from 6 and set $x^0=y^0$ in the exponents, we have to remember that the scalar product in flat space is

$$p\cdot(x-y) = p^0\left(x^0-y^0\right)-\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) \tag{10}$$

Using this, we have

$$[\phi(\boldsymbol{x},t),\phi(\boldsymbol{y},t)] = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] \tag{11}$$

That is, the signs in the exponents are reversed from Coleman's eqn 3.60. However, the conclusion is still valid, since the integrand is an odd function of $\boldsymbol{p}$, so it still does evaluate to zero.

A similar mistake is made in the first line of eqn 3.61. From 6, we have, after taking the time derivative:

$$\left[\dot{\phi}(x),\phi(y)\right]=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[-i\omega_{\boldsymbol{p}}e^{-ip\cdot(x-y)}-i\omega_{\boldsymbol{p}}e^{ip\cdot(x-y)}\right] \quad (12)$$

Setting $x^0=y^0$ gives us

$$\left[\dot{\phi}(\boldsymbol{x},t),\phi(\boldsymbol{y},t)\right]=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[-i\omega_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-i\omega_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] \quad (13)$$

$$=-i\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2)}\left[e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right] \quad (14)$$

and again, the signs in the exponents are the opposite of eqn 3.61. However, this reduces to the same as the second line in eqn 3.61, so the conclusion is still valid:

$$\left[\dot{\phi}(\boldsymbol{x},t),\phi(\boldsymbol{y},t)\right]=-i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \quad (15)$$

Anyway, as I say, I'm mystified as to the point of all this, so if anyone has any comments, please do let me know.

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