FUNCTIONAL DERIVATIVE - A 4-DIMENSIONAL EXAMPLE

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Now for a more involved example of a functional derivative. We define the functional as

$$Z_0[J] = \exp\left[-\frac{1}{2}\int d^4x\, d^4y\; J(x)\Delta(x-y)J(y)\right] \tag{1}$$

where $\Delta(x) = \Delta(-x)$.

Here, we're taking x and y to be four-dimensional vectors (such as might be used in relativity to represent space-time). To find the functional derivative $\frac{\delta Z_0[J]}{\delta J(z_1)}$ we can use the generalization of the functional derivative that we defined earlier. That is, we have

$$\frac{\delta Z_0[J]}{\delta J(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right] - Z_0[J(x)]\right) \tag{2}$$

where $\delta^{(4)}$ is the four-dimensional delta function

$$\delta^{(4)}(x) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta(x_4) \tag{3}$$

To find the derivative, we first find $Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right]$ to first order in ϵ.

$$
\begin{aligned}
Z_0\left[J(x)+\epsilon\delta^{(4)}(z-x)\right] &= \exp\left[-\frac{1}{2}\int d^4x d^4y \left(J(x)+\epsilon\delta^{(4)}(z-x)\right)\times\right.\\
&\quad\left.\Delta(x-y)\left(J(y)+\epsilon\delta^{(4)}(z-y)\right)\right] \qquad (4)\\
&\approx \exp\left[-\frac{1}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right]\times\\
&\quad \exp\left[-\frac{\epsilon}{2}\int d^4x d^4y \times\right.\\
&\quad\left.\left(\delta^{(4)}(z-x)\Delta(x-y)J(y)+\delta^{(4)}(z-y)\Delta(x-y)J(x)\right)\right] \qquad (5)\\
&= Z_0[J]\exp\left[-\frac{\epsilon}{2}\int d^4x d^4y\left(\delta^{(4)}(z-x)\Delta(x-y)J(y)+\right.\right.\\
&\quad\left.\left.\delta^{(4)}(z-y)\Delta(x-y)J(x)\right)\right] \qquad (6)\\
&= Z_0[J]\exp\left[-\frac{\epsilon}{2}\left(\int d^4y\Delta(z-y)J(y)+\int d^4x\Delta(x-z)J(x)\right)\right] \qquad (7)\\
&= Z_0[J]\exp\left[-\epsilon\int d^4y\Delta(z-y)J(y)\right] \qquad (8)
\end{aligned}
$$

where we renamed the integration variable x to y and used $\Delta(x)=\Delta(-x)$ in 7.

Since the exponent in the last line is small, we can expand it in a Taylor series and keep only up to the first order term:

$$
Z_0[J]\exp\left[-\epsilon\int d^4y\Delta(z-y)J(y)\right]\approx Z_0[J]\left(1-\epsilon\int d^4y\Delta(z-y)J(y)\right) \qquad (9)
$$

Plugging this back into 2 and taking the limit we get

$$
\frac{\delta Z_0[J]}{\delta J(z)} = -Z_0[J]\int d^4y\Delta(z-y)J(y) \qquad (10)
$$