

## GENERAL INFINITESIMAL LORENTZ TRANSFORMATION

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The Lorentz transformation along the  $x^1$  axis is

$$\mathbf{\Lambda}(\beta^1) = \begin{bmatrix} \gamma^1 & \beta^1 \gamma^1 & 0 & 0 \\ \beta^1 \gamma^1 & \gamma^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where as usual,  $\beta^1$  is the velocity in the  $x^1$  direction (using natural units so that  $c = 1$ ) and

$$\gamma^1 = \frac{1}{\sqrt{1 - (\beta^1)^2}} \quad (2)$$

For an infinitesimal boost  $\beta^1 = v^1 \ll 1$ , we have  $\gamma^1 \approx 1$  (to first order in  $v^1$ ) so

$$\Lambda_{\nu}^{\mu} = \begin{bmatrix} 1 & v^1 & 0 & 0 \\ v^1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

For an infinitesimal boost  $v^2$  along the  $x^2$  axis, we have

$$\Lambda_{\sigma}^{\rho} = \begin{bmatrix} 1 & 0 & v^2 & 0 \\ 0 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

A combined Lorentz transformation with boosts along the  $x^1$  and  $x^2$  axes is obtained by multiplying these two transformations (in either order; in the case of infinitesimal transformations, the matrices commute), to get

$$\Lambda_{\nu}^{\mu} = \begin{bmatrix} 1 & v^1 & v^2 & 0 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

Finally, we can add in an infinitesimal boost  $v^3$  in the  $x^3$  direction to get

$$\Lambda_{\nu}^{\mu} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

The rotation matrix for a rotation by angle  $\alpha^3$  about the  $x^3$  axis is

$$R^3(\alpha^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha^3 & -\sin \alpha^3 & 0 \\ 0 & \sin \alpha^3 & \cos \alpha^3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

I think that L&B have the wrong sign for the angles of rotation.

For an infinitesimal rotation where  $\alpha^3 = \theta^3 \ll 1$ , this becomes

$$R^3(\theta^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta^3 & 0 \\ 0 & \theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Similarly, for infinitesimal rotations about the other two axes we have

$$R^2(\theta^2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & -\theta^2 & 0 & 1 \end{bmatrix} \quad (9)$$

$$R^1(\theta^1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta^1 \\ 0 & 0 & \theta^1 & 1 \end{bmatrix} \quad (10)$$

Remember that the sign for a rotation about  $x^2$  is opposite to the other two axes.

As with infinitesimal boosts, we can multiply these matrices together in any order to get the overall rotation matrix for a combination of infinitesimal rotations about all three axes:

$$R^\mu_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta^3 & \theta^2 \\ 0 & \theta^3 & 1 & -\theta^1 \\ 0 & -\theta^2 & \theta^1 & 1 \end{bmatrix} \quad (11)$$

Finally, if we multiply 6 by 11 and keep only first order terms (so we ignore terms like  $v^1\theta^2$  since they are products of two infinitesimals), we get the general Lorentz transformation for a combination of infinitesimal boosts and rotations.

$$\Lambda^\mu_\nu = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 1 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 1 \end{bmatrix} \quad (12)$$

We can write this as  $\Lambda = I + \omega$  where the infinitesimal matrix is

$$\omega^\mu_\nu = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix} \quad (13)$$

We can raise or lower indices on  $\omega^\mu_\nu$  by using the metric tensor  $g_{\mu\nu}$ , where  $g_{00} = g^{00} = +1$  and  $g_{ii} = g^{ii} = -1$ . Thus we have

$$\omega^{\mu\nu} = \omega^\mu_\lambda g^{\lambda\nu} \quad (14)$$

$$= \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \quad (15)$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega^\lambda_\nu \quad (16)$$

$$= \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & \theta^3 & -\theta^2 \\ -v^2 & -\theta^3 & 0 & \theta^1 \\ -v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \quad (17)$$

We can see by inspection that both  $\omega^{\mu\nu}$  and  $\omega_{\mu\nu}$  are antisymmetric, and that

$$v^i = -\omega^{0i} = \omega_{0i} \quad (18)$$

$$\theta^i = \frac{1}{2} \varepsilon^{ijk} \omega^{jk} \quad (19)$$

Again, I think L&B have the sign wrong in their equation 9.58.

PINGBACKS

Pingback: Poincaré group transformations