GENERALIZED COMMUTATOR

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Creation and annihilation operators obey commutation relations for bosons and anticommutation relations for fermions. We can represent both these relations by using a generalized commutator defined by

$$[A,B]_\zeta \equiv AB - \zeta BA \tag{1}$$

for operators A and B, and the parameter $\zeta = +1$ for bosons and $\zeta = -1$ for fermions. Using this notation, we have

$$[A,B]_\zeta = [A,B] \tag{2}$$

for bosons and

$$[A,B]_\zeta = \{A,B\} \tag{3}$$

for fermions. The commutators for fields $\psi(\mathbf{x})$ are

$$\left[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})\right]_\zeta = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{4}$$

$$[\psi(\mathbf{x}),\psi(\mathbf{y})]_\zeta = 0 \tag{5}$$

$$\left[\psi^\dagger(\mathbf{x}),\psi^\dagger(\mathbf{y})\right]_\zeta = 0 \tag{6}$$

Lancaster &Blundell define the commutator without the factor of $(2\pi)^3$ that appears in other books, but this factor turns up elsewhere so everything works out the same in the end.

Using L&B's definition of V_{wrong} (an incorrect form for the potential energy) for a two-particle interaction:

$$V_{\text{wrong}} = \frac{1}{2}\int d^3x\, d^3y\, V(\mathbf{x},\mathbf{y})\,\rho(\mathbf{y})\,\rho(\mathbf{x}) \tag{7}$$

where the density operators are given by

$$\rho(\mathbf{x})\,\rho(\mathbf{y}) = \psi^\dagger(\mathbf{x})\,\psi(\mathbf{x})\,\psi^\dagger(\mathbf{y})\,\psi(\mathbf{y}) \tag{8}$$

we can write V_{wrong} using the generalized commutators as follows.

$$\rho(\mathbf{x})\,\rho(\mathbf{y}) = \zeta\psi^{\dagger}(\mathbf{x})\,\psi^{\dagger}(\mathbf{y})\,\psi(\mathbf{x})\,\psi(\mathbf{y}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\,\psi^{\dagger}(\mathbf{x})\,\psi(\mathbf{y}) \tag{9}$$

$$= \zeta^{2}\psi^{\dagger}(\mathbf{x})\,\psi^{\dagger}(\mathbf{y})\,\psi(\mathbf{y})\,\psi(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\,\psi^{\dagger}(\mathbf{x})\,\psi(\mathbf{y}) \tag{10}$$

In the first line, we swapped middle two terms in 8 using 4, and in the last line, we swapped the last two terms $\psi(\mathbf{x})\,\psi(\mathbf{y})$ using 5. Since $\zeta^2 = 1$ for both bosons and fermions, the final result is the same for both:

$$\rho(\mathbf{x})\,\rho(\mathbf{y}) = \psi^{\dagger}(\mathbf{x})\,\psi^{\dagger}(\mathbf{y})\,\psi(\mathbf{y})\,\psi(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\,\psi^{\dagger}(\mathbf{x})\,\psi(\mathbf{y}) \tag{11}$$

This contains an extra self-energy term (the term containing the delta function), which can be avoided by using normal ordering, that is, placing all creation operators to the left of all annihilation operators.