GROUND STATE ENERGY IN A ONE-FIELD MODEL WITH STATIC SOURCE

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We look again at the one-field model with a static source. The interaction hamiltonian is

$$\mathscr{H}_I = g\phi(x)\rho(\boldsymbol{x}) \tag{1}$$

where $\phi(x)$ is the usual scalar field operator, $\rho(\boldsymbol{x})$ is the static source (note that it depends on only the position $\boldsymbol{x}$ and not the time), and g is a parameter governing the strength of the interaction. The model multiplied $\mathscr{H}_I$ by an adiabatic function $f(t)$ which slowly switched on the source and, after a time T, slowly switched it off again.

Using this model, we found that the vacuum-to-vacuum matrix element $\langle 0|S|0\rangle$ picked up a phase that depends on T and the ground state energy E_0, which is physically unrealistic, since we wish to let $T\to\infty$ to regain our model with a static source. Coleman solved this problem by introducing a counterterm a, so that

$$H_I = \left[gf(t)\int d^3\boldsymbol{x}\phi(x)\rho(\boldsymbol{x})\right] - af(t) \tag{2}$$

This essentially cancels out the phase by renormalizing the ground state energy so that it remains at zero even when the interaction is switched on. That is, we find that

$$a = E_0 \tag{3}$$

We would like to find the actual value of a and thus of E_0. We can do this by considering the Wick expansion of the S-matrix, which comes out to

$$S = :\exp\sum_{r=1}^{\infty}\left(\frac{O\left(D_r^{(c)}\right)}{S\left(D_r^{(c)}\right)}\right): \tag{4}$$

FIGURE 1. The two connected diagrams.

See the earlier post for a full explanation. As a reminder, $O\left(D_r^{(c)}\right)$ represents the field operator corresponding to a connected Wick diagram $D_r^{(c)}$, and $S\left(D_r^{(c)}\right)$ is the symmetry factor for that diagram, which is the number of equivalent ways that diagram can be drawn.

For our simple model, there are only two connected diagrams resulting from the field operator $\phi(x)$, as in Fig. 1.

The second diagram is one where the two vertices are contracted, so it is just a number (without any field operators). The term in the Wick expansion arising from this diagram is

$$\frac{O_2}{2!} \tag{5}$$

where the symmetry factor is $S_2 = 2!$ since the two vertices can be swapped to give the same diagram.

The first operator, with one open-ended line, is what we dealt with to find the S-matrix in the earlier post, and is represented by O_1 with symmetry factor 1 (since there is only one vertex, it can't be permuted with anything). As it does contain field operators, it makes no contribution to $\langle 0|S|0\rangle$, since the creation and annihilation operators will give zero when operating on either $\langle 0|$ or on $|0\rangle$.

Due to the counterterm a, there is a third diagram, which Coleman represents by a single $\times$, without any edges. The term in the Wick expansion arising from the counterterm is therefore also just a number without any field operators. This term is called O_3, again with a symmetry factor of 1.

Only O_2 and O_3 can contribute to the ground state energy, since they are the only terms that consist only of numbers, without any field operators. Since the whole purpose of introducing the counterterm was to force the vacuum-to-vacuum matrix element to satisfy

$$\langle 0|S|0\rangle = 1 \tag{6}$$

we must have

$$S = \exp\left(\frac{O_2}{2!} + O_3\right) = 1 \tag{7}$$

or, in other words

$$\frac{O_2}{2!} + O_3 = 0 \tag{8}$$

To find E_0, we therefore need to evaluate either O_2 or O_3. O_3 has the simplest form:

$$O_3 = ia \int dt\ f(t) \tag{9}$$

but of course, since a is what we're trying to find, this expression isn't much use. We therefore need to find O_2.

O_2 consists of a single contraction, which we can evaluate in the usual way:

$$\overbrace{\phi(x_1)\phi}(x_2) = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)} \frac{i}{p^2-\mu^2+i\epsilon} \tag{10}$$

From here, Coleman works out O_2 in equations 9.29 through 9.36. This involves taking the Fourier transforms of ρ and f, and then taking the limit as $T \to \infty$. His derivation is fairly clear, so I won't reproduce it here. The final result is his equation 9.36:

$$a = E_0 = -\frac{g^2}{2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{|\tilde{\rho}(\boldsymbol{p})|^2}{|\boldsymbol{p}|^2+\mu^2} \tag{11}$$

As we haven't specified the actual form for the source $\rho(\boldsymbol{x})$, this is as far as we can go.

Coleman then transforms the result 11 so that it's an integral over position rather than momentum. To do this we need to note that

$$\tilde{\rho}(\boldsymbol{p}) = \int d^3\boldsymbol{x}\ e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \rho(\boldsymbol{x}) \tag{12}$$

Therefore, assuming $\rho(\boldsymbol{x})$ is a real function

$$\tilde{\rho}(-\boldsymbol{p}) = \int d^3\boldsymbol{x}\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \rho(\boldsymbol{x}) \tag{13}$$

$$= \tilde{\rho}(\boldsymbol{p})^* \tag{14}$$

If we define a potential term as

$$V(\boldsymbol{x}) \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{|\boldsymbol{p}|^2+\mu^2} \tag{15}$$

then from 11

$$E_0 = -\frac{g^2}{2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\tilde{\rho}(\boldsymbol{p})\,\tilde{\rho}(-\boldsymbol{p})}{|\boldsymbol{p}|^2+\mu^2} \tag{16}$$

$$= -\frac{g^2}{2(2\pi)^3}\int d^3\boldsymbol{p}\int d^3\boldsymbol{x}\; e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\rho(\boldsymbol{x})\int d^3\boldsymbol{y}\; e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}\rho(\boldsymbol{y})\,\frac{1}{|\boldsymbol{p}|^2+\mu^2} \tag{17}$$

$$= -\frac{g^2}{2(2\pi)^3}\int d^3\boldsymbol{x}\; d^3\boldsymbol{y}\rho(\boldsymbol{x})\,\rho(\boldsymbol{y})\,\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{|\boldsymbol{p}|^2+\mu^2} \tag{18}$$

$$= -\frac{g^2}{2}\int d^3\boldsymbol{x}\; d^3\boldsymbol{y}\rho(\boldsymbol{x})\,V(\boldsymbol{x}-\boldsymbol{y})\,\rho(\boldsymbol{y}) \tag{19}$$

The advantage of this form is that the potential V in 15 doesn't depend on ρ, so it is an integral that we can actually evaluate. Coleman does this for $V(\boldsymbol{x})$ using contour integration in equations 9.40 and 9.41. In this case, the contour consists of the real axis and a semicircle in the upper half plane. The integral is nice in the sense that there are no poles on the real axis, so we can use Cauchy's theorem directly without having to worry about fiddly bits around little semicircles on the real axis. The result is

$$V(r) = \frac{e^{-\mu r}}{4\pi r} \tag{20}$$

where

$$r = |\boldsymbol{x}| \tag{21}$$

This is known as the Yukawa potential. At small distances, it is similar to the Coulomb potential for a point charge, but for larger distances it falls off exponentially so it is essentially a short range force.