

## HAMILTON'S EQUATIONS AND POISSON BRACKETS

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Here we look at another way of writing the equations of motion for a system of particles. We've seen that the Euler-Lagrange equation is derived from the principle of least action using the calculus of variations. The equation for a single particle in one dimension is

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \quad (1)$$

where  $L$  is the Lagrangian

$$L = L(q, \dot{q}) = T - V \quad (2)$$

We can generalize this to a system with  $d$  degrees of freedom (the number of degrees of freedom is the number of particles multiplied by the number of dimensions) as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad (3)$$

where  $i = 1, \dots, d$ . Solving this system of differential equations gives the particle trajectories as functions of time.

The Euler-Lagrange equations can be put into a different form by means of a *Legendre transformation* as follows. We define the *conjugate momenta* as

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \quad (4)$$

We also define the *Hamiltonian*

$$H \equiv \sum_k p_k \dot{q}_k - L \quad (5)$$

Taking derivatives of  $H$  gives (treating  $p_k$  and  $q_k$  as the independent variables):

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_k p_k \frac{\partial \dot{q}_k}{\partial p_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial p_i} \quad (6)$$

$$= \dot{q}_i \quad (7)$$

where we used 4 to cancel off the last two sums.

Similarly, we get

$$\frac{\partial H}{\partial q_i} = \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_k} - \frac{\partial L}{\partial q_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial q_i} \quad (8)$$

$$= -\frac{\partial L}{\partial q_i} \quad (9)$$

Comparing this with 4 and 3 we see that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} - \frac{d}{dt} p_i \quad (10)$$

$$= 0 \quad (11)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (12)$$

where the second line is due to applying the Euler-Lagrange equation 1 to the LHS of the first line.

We thus get Hamilton's equations which are equivalent to the Euler-Lagrange equations:

Hamilton's equations

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}} \quad (13)$$

For a general function  $u(q_i, p_i, t)$  of the generalized coordinates  $q_i$ , conjugate momenta  $p_i$  and time  $t$ , its time derivative is

$$\frac{du}{dt} = \sum_k \frac{\partial u}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial u}{\partial p_k} \dot{p}_k + \frac{\partial u}{\partial t} \quad (14)$$

$$= \sum_k \left( \frac{\partial u}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial u}{\partial t} \quad (15)$$

The sum in the last line is called the *Poisson bracket* and written as

$$\{u, H\} \equiv \sum_k \left( \frac{\partial u}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (16)$$

Hamilton's equations can be written in terms of Poisson brackets. Remember that the independent variables are  $p_i$  and  $q_i$ , so that the derivatives obey:

$$\begin{aligned}
\frac{\partial p_j}{\partial p_k} &= \delta_{jk} \\
\frac{\partial q_j}{\partial q_k} &= \delta_{jk} \\
\frac{\partial p_j}{\partial q_k} &= 0 \\
\frac{\partial q_j}{\partial p_k} &= 0
\end{aligned} \tag{17}$$

We therefore have

$$\dot{p}_i = \{p_i, H\} \tag{18}$$

$$= \sum_k \left( \frac{\partial p_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \tag{19}$$

$$= -\frac{\partial H}{\partial q_i} \tag{20}$$

$$\dot{q}_i = \{q_i, H\} \tag{21}$$

$$= \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \tag{22}$$

$$= \frac{\partial H}{\partial p_i} \tag{23}$$

Finally, we'll have a look at the Poisson brackets for conjugate variables:

$$\{q_i, p_j\} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \tag{24}$$

$$= \delta_{ij} \tag{25}$$

$$\{q_i, q_j\} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) \tag{26}$$

$$= 0 \tag{27}$$

$$\{p_i, p_j\} = \sum_k \left( \frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \tag{28}$$

$$= 0 \tag{29}$$

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