HAMILTON'S EQUATIONS AND POISSON BRACKETS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 30 Dec 2020.

Here we look at another way of writing the equations of motion for a system of particles. We've seen that the Euler-Lagrange equation is derived from the principle of least action using the calculus of variations. The equation for a single particle in one dimension is

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}=0 \tag{1}$$

where L is the Lagrangian

$$L=L\left(q,\dot{q}\right)=T-V \tag{2}$$

We can generalize this to a system with d degrees of freedom (the number of degrees of freedom is the number of particles multiplied by the number of dimensions) as

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}=0 \tag{3}$$

where $i=1,\ldots,d$. Solving this system of differential equations gives the particle trajectories as functions of time.

The Euler-Lagrange equations can be put into a different form by means of a *Legendre transformation* as follows. We define the *conjugate momenta* as

$$p_k\equiv\frac{\partial L}{\partial\dot{q}_k} \tag{4}$$

We also define the *Hamiltonian*

$$H\equiv\sum_k p_k\dot{q}_k-L \tag{5}$$

Taking derivatives of H gives (treating p_k and q_k as the independent variables):

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}_i+\sum_k p_k\frac{\partial\dot{q}_k}{\partial p_i}-\sum_k\frac{\partial L}{\partial\dot{q}_k}\frac{\partial\dot{q}_k}{\partial p_i} &(6)\\ &= \dot{q}_i &(7)\end{aligned}$$

where we used 4 to cancel off the last two sums.

Similarly, we get

$$\frac{\partial H}{\partial q_i} = \sum_k p_k \frac{\partial \dot{q}_k}{\partial q_k} - \frac{\partial L}{\partial q_i} - \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_i} \tag{8}$$

$$= -\frac{\partial L}{\partial q_i} \tag{9}$$

Comparing this with 4 and 3 we see that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} - \frac{d}{dt}p_i \tag{10}$$

$$= 0 \tag{11}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{12}$$

where the second line is due to applying the Euler-Lagrange equation 1 to the LHS of the first line.

We thus get Hamilton's equations which are equivalent to the Euler-Lagrange equations:

Hamilton's equations

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}} \tag{13}$$

For a general function $u(q_i, p_i, t)$ of the generalized coordinates q_i, conjugate momenta p_i and time t, its time derivative is

$$\frac{du}{dt} = \sum_k \frac{\partial u}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial u}{\partial p_k}\dot{p}_k + \frac{\partial u}{\partial t} \tag{14}$$

$$= \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k} \right) + \frac{\partial u}{\partial t} \tag{15}$$

The sum in the last line is called the *Poisson bracket* and written as

$$\{u, H\} \equiv \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \tag{16}$$

Hamilton's equations can be written in terms of Poisson brackets. Remember that the independent variables are p_i and q_i, so that the derivatives obey:

$$\begin{aligned}\frac{\partial p_j}{\partial p_k} &= \delta_{jk}\\ \frac{\partial q_j}{\partial q_k} &= \delta_{jk}\\ \frac{\partial p_j}{\partial q_k} &= 0\\ \frac{\partial q_j}{\partial p_k} &= 0\end{aligned} \tag{17}$$

We therefore have

$$\dot{p}_i = \{p_i, H\} \tag{18}$$

$$= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \tag{19}$$

$$= -\frac{\partial H}{\partial q_i} \tag{20}$$

$$\dot{q}_i = \{q_i, H\} \tag{21}$$

$$= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \tag{22}$$

$$= \frac{\partial H}{\partial p_i} \tag{23}$$

Finally, we'll have a look at the Poisson brackets for conjugate variables:

$$\{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right) \tag{24}$$

$$= \delta_{ij} \tag{25}$$

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right) \tag{26}$$

$$= 0 \tag{27}$$

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right) \tag{28}$$

$$= 0 \tag{29}$$

PINGBACKS

Pingback: Hamilton's equations for relativistic fields

Pingback: Conditions for a transformation to be canonical

Pingback: Classical particle mechanics

Pingback: Classical field theory

Pingback: Time translation symmetry and conservation of energy

Pingback: Time-dependent Lagrangian

Pingback: Lagrangian for a relativistic free particle

Pingback: Momentum and energy for a free particle in an electromagnetic field

Pingback: Hamilton's equations for relativistic fields - conjugate momentum

Pingback: Klein-Gordon equation for fields - derivation from the Lagrangian

Pingback: Poisson brackets in classical field theory