HAMILTONIAN FOR THE DIRAC EQUATION

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Here we continue with the derivation of the total Dirac Hamiltonian. We've seen that the Hamiltonian density is given as

$$\mathcal{H}_0^{1/2} = -i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi \tag{1}$$

where the general solutions of the Dirac equation are given by

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx} + d_r^{\dagger}(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{2}$$

$$\bar{\psi} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,e^{-ipx} + c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right] \tag{3}$$

The full Hamiltonian is the integral of the density over all space, that is

$$H_0^{1/2} = \int\left[-i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi\right]d^3x \tag{4}$$

From section 4.4.1 in Klauber's book and the previous post, we see that four of the terms in the expansion of this integral (after substituting in 2 and 3) cancel out, leaving us with (implied sum over $j = 1,2,3$ in the first two lines):

$$\begin{aligned}H_0^{1/2} = -&\int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,\gamma^j p^j v_s(\mathbf{p})\,d_s^{\dagger}(\mathbf{p})\,d^3x +\\ &\int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,\gamma^j p^j u_s(\mathbf{p})\,c_s(\mathbf{p})\,d^3x +\\ &\int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,m v_s(\mathbf{p})\,d_s^{\dagger}(\mathbf{p})\,d^3x +\\ &\int\sum_{r,s,\mathbf{p}}\frac{m}{VE_{\mathbf{p}}}c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,m u_s(\mathbf{p})\,c_s(\mathbf{p})\,d^3x\end{aligned} \tag{5}$$

Klauber shows that the sum of the first and third lines gives us

$$\sum_{r,\mathbf{p}} E_{\mathbf{p}} d^{\dagger}(\mathbf{p})\, d_r(\mathbf{p}) - 1 \equiv \sum_{r,\mathbf{p}} E_{\mathbf{p}} \bar{N}_r(\mathbf{p}) - 1 \tag{6}$$

By following similar steps to Klauber's equations 4-70 to 4.73, we can work out the sum of the second and fourth lines, as follows. We can use the inner products of the spinors:

$$u_r^{\dagger}(\mathbf{p})\, u_s(\mathbf{p}) = v_r^{\dagger}(\mathbf{p})\, v_s(\mathbf{p}) = \frac{E_{\mathbf{p}}}{m}\delta_{rs} = \frac{p_0}{m}\delta_{rs} \tag{7}$$

The adjoint spinors are defined as

$$\bar{u}_r = u_r^{\dagger}\gamma^0 \tag{8}$$

so combined with the identity $\gamma^0\gamma^0 = 1$ we have

$$u_r^{\dagger}(\mathbf{p})\, u_s(\mathbf{p}) = \bar{u}_r(\mathbf{p})\,\gamma^0 u_s(\mathbf{p}) \tag{9}$$

$$\bar{u}_r(\mathbf{p})\,\gamma^0 p_0 u_s(\mathbf{p}) = \frac{(p_0)^2}{m}\delta_{rs} \tag{10}$$

$$= \frac{E_{\mathbf{p}}^2}{m}\delta_{rs} \tag{11}$$

The sum of the second and fourth lines of 5 can then be written as

$$\int \sum_{r,s,\mathbf{p}} \frac{m}{VE_{\mathbf{p}}} c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\left[-\gamma^j p_j + m - \gamma^0 p_0 + \gamma^0 p_0\right] u_s(\mathbf{p})\, c_s(\mathbf{p})\, d^3x = \tag{12}$$

$$\int \frac{d^3x}{V} \sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} c_r^{\dagger}(\mathbf{p}) \left[\bar{u}_r(\mathbf{p})\,(-\gamma^{\mu} p_{\mu} + m)\, u_r(\mathbf{p}) + \frac{E_{\mathbf{p}}^2}{m}\right] c_r(\mathbf{p}) \tag{13}$$

Using Klauber's equation B4-3.3:

$$(\not{p} - m)\, u_r(\mathbf{p}) = 0 \tag{14}$$

and the fact that

$$\int d^3x = V \tag{15}$$

that is, the integral over the entire volume is just the volume V itself, we see that 13 reduces to

$$\sum_{r,\mathbf{p}} \frac{m}{E_{\mathbf{p}}} c_r^{\dagger}(\mathbf{p}) \frac{E_{\mathbf{p}}^2}{m} c_r(\mathbf{p}) = \sum_{r,\mathbf{p}} E_{\mathbf{p}} c_r^{\dagger}(\mathbf{p}) c_r(\mathbf{p}) \tag{16}$$

$$\equiv \sum_{r,\mathbf{p}} E_{\mathbf{p}} N_r(\mathbf{p}) \tag{17}$$

Combining this result with 6 we get the final Hamiltonian as

$$H_0^{1/2} = \sum_{r,\mathbf{p}} E_{\mathbf{p}} \left(N_r(\mathbf{p}) - \frac{1}{2} + \bar{N}_r(\mathbf{p}) - \frac{1}{2} \right) \tag{18}$$

The operators are interpreted as number operators so that $N_r(\mathbf{p})$ is the operator whose eigenvalue is the number of particles with spin r and momentum $\mathbf{p}$, and $\bar{N}_r(\mathbf{p})$ is the operator whose eigenvalue is the number of antiparticles with spin r and momentum $\mathbf{p}$. The two $\frac{1}{2}$ terms give rise to an infinite negative vacuum energy.

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