HARMONIC OSCILLATOR TO FOCK SPACE

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One of the central problems in relativistic quantum theory is the representation of many-particle systems. Because of relativity's mass-energy equivalence, any quantum system with a high enough energy can result in the creation of new particles. This process is not possible in non-relativistic quantum theory.

The main approach is known as *occupation number representation*. If we first imagine a finite system in a box, then the allowable energies and momenta are quantized into a discrete set (as in the infinite square well problem in non-relativistic theory). Rather than describe a system with many particles by listing each particle and its associated momentum, we can just say how many particles there are with each momentum $\boldsymbol{p}$, and call this $N(\boldsymbol{p})$. This implies that Bose statistics are being used, in which all particles with the same momentum are identical. In this case, we can write the total energy H and total momentum $\boldsymbol{P}$ as

$$H = \sum_{\boldsymbol{p}} \omega_{\boldsymbol{p}} N(\boldsymbol{p}) \tag{1}$$

$$\boldsymbol{P} = \sum_{\boldsymbol{p}} \boldsymbol{p} N(\boldsymbol{p}) \tag{2}$$

At this point, we note a formal similarity with the harmonic oscillator. A quantum harmonic oscillator has a discrete set of possible energy levels, with any two adjacent energies separated by $\hbar\omega_{\boldsymbol{p}}$, or just $\omega_{\boldsymbol{p}}$ if we set $\hbar = 1$. Thus a given oscillator with energy level separation $\omega_{\boldsymbol{p}}$ can be in any state from the ground state (which is $\frac{1}{2}\omega_{\boldsymbol{p}}$ for a proper harmonic oscillator, but we can 'renormalize' this so the ground state is zero) up to any number $N(\boldsymbol{p})$ times $\omega_{\boldsymbol{p}}$. In other words, each term in the sum 1 corresponds to an oscillator in the energy state $\omega_{\boldsymbol{p}} N(\boldsymbol{p})$.

We originally worked out the details of the quantum oscillator using an algebraic approach. We introduced the raising operator a_+ which raised the energy of the oscillator by one unit of $\omega_{\boldsymbol{p}}$, and the lowering operator a_- which lowered the energy by the same amount. The hamiltonian can be expressed in terms of a_+ and a_- (again, neglecting the ground state energy) as

$$H_{\boldsymbol{p}} = \omega_{\boldsymbol{p}} a_+ a_- \tag{3}$$

which leads us to the interpretation of a_+a_- as a number operator which counts the number of energy quanta possessed by the oscillator.

The operators a_+ and a_- were defined originally in terms of the position and momentum operators in quantum theory, and from these definitions we can work out their commutator, which is

$$[a_-, a_+] = 1 \tag{4}$$

If we have a number of uncoupled oscillators, each labelled by its momentum $\boldsymbol{p}$, we can modify the notation a bit so that

$$\begin{aligned} a_- &\equiv a_{\boldsymbol{p}} \\ a_+ &\equiv a_{\boldsymbol{p}}^\dagger \end{aligned} \tag{5}$$

Since the oscillators are uncoupled, an oscillator with one momentum $\boldsymbol{p}$ has no effect on an oscillator with another momentum $\boldsymbol{p}'$, so the corresponding operators must commute. This gives us the set of commutators in the form

$$\begin{aligned} \left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger\right] &= \delta_{\boldsymbol{p}\boldsymbol{p}'} \\ \left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] &= \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{p}'}^\dagger\right] = 0 \end{aligned} \tag{6}$$

The last line follows from the fact that if $\boldsymbol{p} = \boldsymbol{p}'$, any operator commutes with itself.

Now we come to the key point. We use the formal similarity between the occupation number representation for a many particle theory and the notation for a collection of uncoupled harmonic oscillators to propose that we can represent a many particle system by interpreting $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^\dagger$ as operators that annihilate and create *particles* with energy $\omega_{\boldsymbol{p}}$. In the harmonic oscillator, the lowering and raising operators lower or raise the energy level of a single oscillator by one unit, while in the many-particle theory, they annihilate or create a single particle.

If we make the transition from a finite box to infinite space, then the range of allowable momenta becomes continuous, and we replace the commutators with

$$\boxed{\begin{aligned}\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right] &= \delta\left(\boldsymbol{p}-\boldsymbol{p}'\right)\\ \left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}\right] = \left[a_{\boldsymbol{p}}^{\dagger},a_{\boldsymbol{p}'}^{\dagger}\right] &= 0\end{aligned}} \tag{7}$$

We haven't yet created any actual quantum *fields*. Rather, we propose that the fields will be linear combinations of the $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ operators.

A multi-particle quantum state is represented in what is known as *Fock space*, named after Vladimir Fock (1898 - 1974), a Soviet physicist. A state in Fock space is written using the usual ket notation as

multi-particle Fock state

$$\left|\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_n\right\rangle \tag{8}$$

where there are n particles with the momenta given in the ket.

The normalization is, for a two-particle state

$$\begin{aligned}\left\langle\boldsymbol{p}_1',\boldsymbol{p}_2'\right|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle = \delta\left(\boldsymbol{p}_1-\boldsymbol{p}_1'\right)\delta\left(\boldsymbol{p}_2-\boldsymbol{p}_2'\right)+\\ \delta\left(\boldsymbol{p}_1-\boldsymbol{p}_2'\right)\delta\left(\boldsymbol{p}_2-\boldsymbol{p}_1'\right)\end{aligned} \tag{9}$$

For states with more particles, we have a sum of products of delta functions, with each term in the sum being one permutation of the combination of states in the bra and states in the ket.

The ground state is written as

vacuum state

$$|0\rangle \tag{10}$$

and is called the *vacuum state*. Note that this *not* the same thing as a mathematical zero; rather it represents a real, physical state with no particles in it. It is taken to be a state with zero energy and zero momentum, and is a Lorentz invariant state, so a Lorentz transformation of $|0\rangle$ just gives us $|0\rangle$ back again.

Using the same normalization as with the harmonic oscillator

$$\begin{aligned}a_{\boldsymbol{p}}^{\dagger}|0\rangle &= \sqrt{0+1}\,|\boldsymbol{p}\rangle = |\boldsymbol{p}\rangle\\ a_{\boldsymbol{p}}|\boldsymbol{p}\rangle &= \sqrt{1}\,|0\rangle = |0\rangle\end{aligned} \tag{11}$$

Note that

$$a_{\boldsymbol{p}}|0\rangle = 0 \tag{12}$$

where the 0 on the RHS is a mathematical zero, and not a state in Fock space.

These creation and annihilation operators are defined in 3-momentum space. We can define relativistically normalized versions of these operators in 4-momentum space as

$$\begin{aligned}\alpha^{\dagger}(p) &\equiv (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}^{\dagger} \\ \alpha(p) &\equiv (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}\end{aligned} \tag{13}$$

The action of $\alpha^{\dagger}$ on the vacuum is then

$$\alpha^{\dagger}(p)|0\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}^{\dagger}|0\rangle \tag{14}$$

$$= (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}|\boldsymbol{p}\rangle = |p\rangle \tag{15}$$

where in the last line, we have the 4-momentum state $|p\rangle$ which has 3-momentum $\boldsymbol{p}$ and $p^0 = \omega_{\boldsymbol{p}}$. As we saw when finding the relativistic invariant integration measure, we can integrate $|p\rangle\langle p|$ in 4-momentum space over the mass shell to get the unit operator.

$$1 = \frac{1}{(2\pi)^3}\int d^4p\,\delta\left(p^2-\mu^2\right)\theta\left(p^0\right)|p\rangle\langle p| \tag{16}$$

$$= \frac{1}{(2\pi)^3}\int\frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}(2\pi)^3\,2\omega_{\boldsymbol{p}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \tag{17}$$

$$= \int d^3\boldsymbol{p}\;|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \tag{18}$$

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