INTERACTION PICTURE EQUATIONS OF MOTION

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Our earlier post on the interaction picture followed Coleman's approach. The treatment by Klauber is a bit different, and I think it highlights the main features somewhat better, so we'll have a look at it here.

Klauber's treatment deals with the full Hamiltonian (not just the Hamiltonian density). The interaction picture is used (as its name implies) when we're dealing with interacting fields, and not just with free fields. The idea is that we split the Hamiltonian into a free field part and an interaction part, so we have

$$H^S = H_0^S + H_I^S \tag{1}$$

The placement of superscripts and subscripts is important for understanding the derivations. For quantum states, a subscript indicates what picture we're dealing with, so that $|\Psi\rangle_S$ is a state in the Schrödinger picture (where states depend on time) and $|\Psi\rangle_I$ is a state in the interaction picture. In the Heisenberg picture, the states $|\Psi\rangle_H$ do not depend on time.

For all operators, a superscript I, H or S indicates which picture we're referring to. A subscript 0 indicates the free field portion, while a subscript I indicates the interaction portion. In what follows, we take $H_0^S \equiv H_0$, that is, the free portion of the Hamiltonian without a superscript is defined to be the free Hamiltonian in the Schrödinger picture.

For example, in 1, a subscript 0 indicates the Hamiltonian for the free field, and a subscript I indicates the interaction portion of the Hamiltonian. Thus H^S is the full Hamiltonian in the Schrödinger picture, H_0^S is the free field portion and H_I^S is the interaction portion, both in the Schrödinger picture (SP from here on).

The transformation from the SP to the interaction picture (IP) is defined by using the operator

$$U_0 = e^{-iH_0 t} \tag{2}$$

From its definition, U_0 is unitary, so we have

$$U_0^\dagger = U^{-1} = e^{iH_0 t} \tag{3}$$

Note that U_0 depends *only* on the free Hamiltonian H_0. States are transformed by

$$|\Psi\rangle_I = U_0^\dagger |\Psi\rangle_S = e^{iH_0t}|\Psi\rangle_S \tag{4}$$

Operators are transformed by

$$\mathcal{O}^I = U_0^\dagger \mathcal{O}^S U_0 = e^{iH_0t}\mathcal{O}^S e^{-iH_0t} \tag{5}$$

In particular, the free and interaction portions of the Hamiltonian in the SP transform as follows.

$$H_0^I = e^{iH_0t}H_0e^{-iH_0t} \tag{6}$$

Since the exponential of an operator, as in e^{iH_0t}, is defined to be the power series of that operator, and because H_0 commutes with itself, we have

$$\begin{aligned} H_0^I &= e^{iH_0t}e^{-iH_0t}H_0 & (7)\\ &= H_0 & (8) \end{aligned}$$

That is, the free Hamiltonian is the same in both the SP and the IP.

For the interaction part, we have

$$H_I^I = e^{iH_0t}H_I^S e^{-iH_0t} \tag{9}$$

(remember that H_I^S is the *interaction* part (subscript) of the Hamiltonian in the SP).

In general, H_I^S does *not* commute with H_0, so we can't change the order of the factors on the RHS of 9. As a result, we can't bring the two exponentials next to each other, so we can't cancel them off. That is, in general

$$H_I^I \neq H_I^S \tag{10}$$

The full interaction Hamiltonian is therefore

$$H^I = H_0 + H_I^I \tag{11}$$

The equation of motion for the IP states $|\Psi\rangle_I$ can be found by starting with the equation of motion in the SP, which is given by the Schrödinger equation

$$i\frac{d}{dt}|\Psi\rangle_S = H^S|\Psi\rangle_S \tag{12}$$

We can expand this equation as follows. The LHS is

$$i\frac{d}{dt}|\Psi\rangle_S = i\frac{d}{dt}\left(U_0|\Psi\rangle_I\right) \tag{13}$$
$$= i\left[\left(-iH_0U_0|\Psi\rangle_I + U_0\frac{d}{dt}|\Psi\rangle_I\right)\right] \tag{14}$$
$$= H_0U_0|\Psi\rangle_I + iU_0\frac{d}{dt}|\Psi\rangle_I \tag{15}$$

The RHS of 12 is

$$H^S|\Psi\rangle_S = \left(H_0 + H_I^S\right)U_0|\Psi\rangle_I \tag{16}$$
$$= \left(H_0 + U_0H_I^IU_0^\dagger\right)U_0|\Psi\rangle_I \tag{17}$$
$$= H_0U_0|\Psi\rangle_I + U_0H_I^I|\Psi\rangle_I \tag{18}$$

Equating 15 and 18 we cancel off the common term $H_0U_0|\Psi\rangle_I$ to get

$$iU_0\frac{d}{dt}|\Psi\rangle_I = U_0H_I^I|\Psi\rangle_I \tag{19}$$

We now multiply this on the left by $U_0^\dagger$ to get the equation of motion for IP states:

$$\boxed{i\frac{d}{dt}|\Psi\rangle_I = H_I^I|\Psi\rangle_I} \tag{20}$$

Equation of motion for IP states

The equation of motion for operators can be found as follows. We start from 5 and use the product rule.

$$\frac{d}{dt}\mathcal{O}^I = \frac{d}{dt}\left(e^{iH_0t}\mathcal{O}^Se^{-iH_0t}\right) \tag{21}$$
$$= iH_0e^{iH_0t}\mathcal{O}^Se^{-iH_0t} + e^{iH_0t}\frac{\partial\mathcal{O}^S}{\partial t}e^{-iH_0t} - e^{iH_0t}\mathcal{O}^Se^{-iH_0t}iH_0 \tag{22}$$

In the SP, it is usual for operators to have no explicit time dependence, so the middle term is zero, as $\frac{\partial\mathcal{O}^S}{\partial t} = 0$. We therefore get

$$\frac{d}{dt}\mathcal{O}^I = iH_0e^{iH_0t}\mathcal{O}^Se^{-iH_0t} - e^{iH_0t}\mathcal{O}^Se^{-iH_0t}iH_0 \tag{23}$$
$$= iH_0\mathcal{O}^I - i\mathcal{O}^IH_0 \tag{24}$$
$$= i\left[H_0, \mathcal{O}^I\right] \tag{25}$$

The equation of motion for IP operators is therefore

Equation of motion for IP operators

$$\boxed{\frac{d}{dt}\mathcal{O}^I = i\left[H_0, \mathcal{O}^I\right]} \tag{26}$$

Note that the commutator involves *only* the **free** Hamiltonian H_0, which is the same on both the SP and IP.

To see the value of the IP, we compare it with the Heisenberg picture (HP). In the HP, the operators are defined as

$$\mathcal{O}^H = U^{\dagger}(t,0)\,\mathcal{O}^S U(t,0) \tag{27}$$

where the time evolution operator is defined as

$$U(t,0) = e^{-iHt} \tag{28}$$

where H here is the *full* Hamiltonian. The Heisenberg equation of motion is then obtained by following the same derivation as above, but with H_0 replaced by H. We get

$$\frac{d}{dt}\mathcal{O}^H = i\left[H, \mathcal{O}^H\right] \tag{29}$$

If we compare this with 26, we see that the equation of motion for an IP operator is the same as that for an HP operator when we use the free Hamiltonian H_0 instead of the full Hamiltonian H. That is, the special case of $H = H_0$ in the HP applies to the *general* case in the IP. This means that we can take the solutions we worked out for free fields in the HP and apply them directly to the general case in the IP.

In particular, this means that the behaviour of the field operators (ϕ in scalar (Klein-Gordon) fields, ψ in Dirac fields and A^{μ} in Maxwell's equations) is the same in the IP as in the HP with free fields.

The catch, of course, is that we have to introduce the interactions *somewhere* in order to get a general solution for a Lagrangian with interaction terms. This is done via the equation of motion for states 20, which is the only place that the interaction portion of the Hamiltonian shows up. In most (almost all) cases, an exact solution of 20 isn't possible, so we need to resort to perturbation theory.

PINGBACKS

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