INTERACTION PICTURE

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The interaction picture (also called the Dirac picture) is the prelude to perturbation theory. It can be a bit tricky to follow the derivations here due to the number of different hamiltonians, but if you take it slowly, it should become clear. A reminder: a subscript S on an operator or state indicates the Schrödinger picture, and a subscript H indicates the Heisenberg picture.

We assume that we have a free time-independent Hamiltonian H_0 and an extra bit H' that could depend on time. That is

$$H = H_0\left(p,q\right) + H'\left(p,q,t\right) \tag{1}$$

In the interaction picture, we define an interaction version of q by modifying the relation between the Schrödinger and Heisenberg position operators, which was:

$$q_H\left(t\right) = U^{\dagger}\left(t,0\right) q_S U\left(t,0\right) \tag{2}$$

We now define a new position operator $q_I\left(t\right)$ (subscript I for 'interaction picture') by taking the evolution operator U to depend *only* on H_0 (so we'll call it U_0):

$$\begin{aligned} q_I\left(t\right) &= U_0^{\dagger}\left(t,0\right) q_S U_0\left(t,0\right) &(3)\\ &= e^{iH_0(p_S,q_S)t} q_S e^{-iH_0(p_S,q_S)t} &(4) \end{aligned}$$

Here we've used the explicit form of the evolution operator

$$U\left(t,t'\right) = e^{-iH(p_S,q_S)(t-t')} \tag{5}$$

with $t' = 0$. Remember that this form is valid only if the hamiltonian H does not depend on time, which is what we're assuming is true for the free hamiltonian H_0.

We also define interaction picture states as

$$\left|\psi\left(t\right)\right\rangle_I = e^{iH_0(p_S,q_S)t}\left|\psi\left(t\right)\right\rangle_S \tag{6}$$

and a general transformation for operators

$$A_I(t) = e^{iH_0(p_S,q_S)t} A_S(t) e^{-iH_0(p_S,q_S)t} \tag{7}$$

These definitions are to ensure that probability and expectation values are conserved. For example, the expectation value of an operator A is the same in both the interaction and Schrödinger pictures:

$$_I\langle\psi(t)|A_I(t)|\psi(t)\rangle_I = _S\left\langle\psi(t)\left|e^{-iH_0(p_S,q_S)t}e^{iH_0(p_S,q_S)t}A_S(t)e^{-iH_0(p_S,q_S)t}e^{iH_0(p_S,q_S)t}\right|\psi(t)\right\rangle_S \tag{8}$$

$$= _S\langle\psi(t)|A_S(t)|\psi(t)\rangle_S \tag{9}$$

A consequence of 7 is that all operators A_I in the interaction picture evolve according to the *free* hamiltonian H_0. This means that, in particular, the creation and annihilation operators in the free field remain as free fields in the interaction picture; the effects of the interaction term in the hamiltonian are felt only in the states. This fact isn't stated explicitly in any of the books I've looked at, yet it is used implicitly in much of what follows.

The main goal of the interaction picture is to derive a differential equation for the interaction state $|\psi(t)\rangle_I$ which in general will not be exactly solvable. We then apply perturbation theory to find an approximate solution. We obtain the differential equation from 6:

$$\frac{d}{dt}|\psi(t)\rangle_I = e^{iH_0(p_S,q_S)t}\left[iH_0(p_S,q_S)|\psi(t)\rangle_S + \frac{d}{dt}|\psi(t)\rangle_S\right] \tag{10}$$

The last derivative on the RHS is obtained by applying the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle_S = H(p_S,q_S,t)|\psi(t)\rangle_S \tag{11}$$

We need to remember that the H in 11 is the *full* Hamiltonian H 1 and not just the free Hamiltonian H_0 that we've been using so far. Using this, Coleman shows in eqn 7.22 and 7.23 that the differential equation is

$$\frac{d}{dt}|\psi(t)\rangle_I = -iH'(p_I,q_I,t)|\psi(t)\rangle_I \tag{12}$$

where

$$H'(p_I,q_I,t) \equiv H_I(t) = e^{iH_0(p_S,q_S)t}H'_S(p_S,q_S,t)e^{-iH_0(p_S,q_S)t} \tag{13}$$

Note also that the position and momentum operators have morphed from (p_S,q_S) in 10 to (p_I,q_I) in 12. This is possible if we assume that the Hamiltonian $H'_S(p_S,q_S,t)$ in 13 can be expanded in a power series in p_S and q_S, so we can then insert factors of $e^{iH_0(p_S,q_S)t}e^{-iH_0(p_S,q_S)t}$ between all pairs of factors of p_S and q_S to convert them to p_I and q_I using 4.

To use perturbation theory, an interaction picture evolution operator $U_I(t,t')$ is introduced by the equation

$$|\psi(t)\rangle_I = U_I(t,t')\,|\psi(t')\rangle_I \tag{14}$$

This evolution operator has the same properties as the original evolution operator introduced in the Schrödinger picture. That is

$$U_I^\dagger(t,t') = U_I^{-1}(t,t') \tag{15}$$

which means the operator U_I is unitary. There is also a composition rule:

$$U_I(t,t'') = U_I(t,t')\,U_I(t',t'') \tag{16}$$

Combining this with 15 we have

$$U_I(t',t) = U_I^{-1}(t,t') \tag{17}$$

The relation between U_I and the original U (the latter of which depends on the full Hamiltonian H) is derived in Coleman's eqn 7.31:

$$U_I(t,0) = e^{iH_0t}U(t,0) = e^{iH_0t}e^{-iHt} \tag{18}$$

Note carefully that the first exponent has the free Hamiltonian H_0 while the second has the full Hamiltonian H.

Coleman develops the properties of U_I in eqns 7.25 through 7.33, ending with the central differential equation for U_I which is to be solved using perturbation theory:

$$\boxed{i\frac{\partial}{\partial t}U_I(t,t') = H_I(t)\,U_I(t,t')} \tag{19}$$

From 6, we get the interpretation of $|\psi(t)\rangle_I$. The Schrödinger state $|\psi(t)\rangle_S$ is the exact state, which evolves according to the full hamiltonian 1. This exact state is what, in an ideal world, we like to find. Turning the definition around, we have

$$|\psi(t)\rangle_S = e^{-iH_0(p_S,q_S)t}\,|\psi(t)\rangle_I \tag{20}$$

Thus if we can find $|\psi(t)\rangle_I$, we can find the exact state $|\psi(t)\rangle_S$ by applying the (known) evolution operator using the free hamiltonian. The problem, of course, is that solving 19 exactly is not usually possible, so we must resort to perturbation theory.

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