

KLEIN-GORDON EQUATION - COMMUTATORS

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The solutions to the Klein-Gordon equation

$$(\square^2 + \mu^2) \phi = 0 \quad (1)$$

have discrete plane-wave solutions of form

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{ikx} \right) \quad (2)$$

and continuous plane-wave solutions of form

$$\phi(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} b^\dagger(\mathbf{k}) e^{ikx} \quad (3)$$

The second quantization postulate is that the Poisson brackets from classical field theory translate into commutators in quantum field theory, so that

$$[\phi^r(\mathbf{x}, t), \pi_s(\mathbf{x}', t)] = i\hbar \delta_s^r \delta^3(\mathbf{x} - \mathbf{x}') \quad (4)$$

$$[\phi^r, \phi^s] = 0 \quad (5)$$

$$[\pi_r, \pi_s] = 0 \quad (6)$$

where the conjugate momentum is, for the free-field scalar Lagrangian density we're using here:

$$\pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} = \dot{\phi}^{r\dagger} \quad (7)$$

In Klauber's section 3.3, he goes through a detailed calculation for the discrete solution 2 to show that applying the commutation relation 4 to this solution results in commutation relations for the a and b operators:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'} \quad (8)$$

To abbreviate the notation a bit, we'll write this as

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger \right] = \left[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger \right] = \delta_{\mathbf{k}\mathbf{k}'} \quad (9)$$

The derivation of the corresponding commutators for the continuous solution 3 is very similar, although a bit lengthy, but it's useful to review the process.

In the derivation, we have two fields: ϕ and ϕ^\dagger and their conjugate momenta π and π^\dagger . From 3 and 7, and the fact that k is a 4-vector:

$$k = (\omega_{\mathbf{k}}, \mathbf{k}) \quad (10)$$

we have

$$\phi(\mathbf{x}, t) = \int \frac{d^3 k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + \int \frac{d^3 k}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}} b_{\mathbf{k}}^\dagger e^{i(\omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} \quad (11)$$

$$\pi(\mathbf{x}', t) = \int \frac{i\omega_{\mathbf{k}'} d^3 k'}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}'}}} a_{\mathbf{k}'}^\dagger e^{i(\omega_{\mathbf{k}'} t - \mathbf{k}' \cdot \mathbf{x}')} + \int \frac{-i\omega_{\mathbf{k}'} d^3 k'}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}'}}} b_{\mathbf{k}'} e^{-i(\omega_{\mathbf{k}'} t - \mathbf{k}' \cdot \mathbf{x}')} \quad (12)$$

I've used different integration variables k and k' for the two equations, since we'll be multiplying them and we need to keep the integration variables separate. If we substitute these two equations into the LHS of 4 and multiply them out, we get a long expression which we can write as

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = \frac{1}{2(2\pi)^3} \int d^3 k \int d^3 k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}}} X \quad (13)$$

where X is

$$X = \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger \right] e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{i(\mathbf{k} \cdot \mathbf{x} - \mathbf{k}' \cdot \mathbf{x}')} - \quad (14)$$

$$\left[a_{\mathbf{k}}, b_{\mathbf{k}'} \right] e^{-i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}})t} e^{i(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')} - \quad (15)$$

$$\left[a_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger \right] e^{i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}})t} e^{-i(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}')} + \quad (16)$$

$$\left[b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}} \right] e^{-i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(\mathbf{k} \cdot \mathbf{x} - \mathbf{k}' \cdot \mathbf{x}')} \quad (17)$$

The delta function on the RHS of 4 can be written as an integral:

$$\delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int d^3 k e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \quad (18)$$

Therefore, we must have

$$\frac{1}{2} \int d^3 k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X = e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \quad (19)$$

The RHS of 19 has no time dependence, so the terms in 15 and 16 must be zero, since they contain a factor $e^{\pm i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}})t}$ which is time-dependent, no matter what \mathbf{k} and \mathbf{k}' are (remember that $\omega_{\mathbf{k}}$ is always positive as it represents a frequency). Therefore

$$[a_{\mathbf{k}}, b_{\mathbf{k}'}] = [a_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger] = 0 \quad (20)$$

The remaining two terms 14 and 17 are independent of time if $\omega_{\mathbf{k}} = \omega_{\mathbf{k}'}$ which implies that $|\mathbf{k}| = |\mathbf{k}'|$. Choosing

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = [b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}') \quad (21)$$

certainly satisfies this condition as well as the original commutator 4, since

$$\frac{1}{2(2\pi)^3} \int d^3 k \int d^3 k' \frac{i\omega_{\mathbf{k}'}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} X = \frac{i}{2(2\pi)^3} \int d^3 k \int d^3 k' \delta^3(\mathbf{k} - \mathbf{k}') \left(e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} + e^{-i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} \right) \quad (22)$$

$$= \frac{i}{2(2\pi)^3} \int d^3 k \left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \right) \quad (23)$$

$$= i\delta^3(\mathbf{x} - \mathbf{x}') \quad (24)$$

It would seem to me that the justification for this choice (rather than a more general condition which gives non-zero commutators for any pair \mathbf{k} and \mathbf{k}' where $|\mathbf{k}| = |\mathbf{k}'|$, and not the more restrictive condition that $\mathbf{k} = \mathbf{k}'$) is that these more general conditions don't satisfy 4.

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