KLEIN-GORDON EQUATION - CONTINUOUS SOLUTIONS

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The Klein-Gordon equation

$$\left(\Box^2+\mu^2\right)\phi=0 \tag{1}$$

has discrete plane-wave solutions of form

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}e^{-ikx}+B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{2}$$

where V is the volume containing the waves. The frequencies $\omega_{\mathbf{k}}$ and wave numbers **k** are constrained by the requirement that the waves must fit within V, so that their amplitudes are zero at the boundary.

A more general solution which allows all frequencies of waves and is not confined to a particular volume is given by

$$\begin{aligned}
\phi(x) &= \int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a(\mathbf{k})e^{-ikx}+\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b^{\dagger}(\mathbf{k})e^{ikx} && (3)\\
&\equiv \phi^{+}+\phi^{-} && (4)\\
\phi^{\dagger}(x) &= \int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b(\mathbf{k})e^{-ikx}+\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a^{\dagger}(\mathbf{k})e^{ikx} && (5)\\
&\equiv \phi^{\dagger+}+\phi^{\dagger-} && (6)
\end{aligned}$$

To verify that these are solutions of the Klein-Gordon equation 1, we note that the operator $\Box^2$ acts only on x which appears only in the exponential factors inside the integrals. Thus we get

$$\left(\square^2+\mu^2\right)\phi=\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a(\mathbf{k})\left((-i)^2k_\mu k^\mu+\mu^2\right)e^{-ikx}+$$

$$\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b^\dagger(\mathbf{k})\left(i^2k_\mu k^\mu+\mu^2\right)e^{ikx} \tag{7}$$

$$=\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a(\mathbf{k})\left(\mu^2-k_\mu k^\mu\right)e^{-ikx}+$$

$$\int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b^\dagger(\mathbf{k})\left(\mu^2-k_\mu k^\mu\right)e^{ikx} \tag{8}$$

As with the discrete case, we can use the relativistic condition that $k_\mu k^\mu = m^2\frac{c^2}{\hbar^2}$ to find that $\mu=\frac{mc}{\hbar}$ (or just m in natural units), so the integral solution does indeed satisfy the Klein-Gordon equation.

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