

## KLEIN-GORDON EQUATION - ORTHONORMALITY OF SOLUTIONS

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The plane wave solutions of the Klein-Gordon equation are

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left( A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \quad (1)$$

We can redefine a couple of terms by introducing

$$\phi_{\mathbf{k},A} \equiv \frac{e^{-ikx}}{\sqrt{V}} \quad (2)$$

$$\phi_{\mathbf{k},B^{\dagger}} \equiv \frac{e^{ikx}}{\sqrt{V}} \quad (3)$$

Then

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( A_{\mathbf{k}} \phi_{\mathbf{k},A} + B_{\mathbf{k}}^{\dagger} \phi_{\mathbf{k},B^{\dagger}} \right) \quad (4)$$

The  $\phi_{\mathbf{k},A}$  and  $\phi_{\mathbf{k},B^{\dagger}}$  are orthonormal functions. We have

$$\int \phi_{\mathbf{k},A}^{\dagger} \phi_{\mathbf{k}',A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x \quad (5)$$

where the integral is over the volume  $V$ , and the wavelengths of the plane waves fit an integral number of times within  $V$ , so that the amplitudes of the waves at the boundaries are all zero. The four-vector  $k$  is defined as

$$k = [\omega_{\mathbf{k}}, \mathbf{k}] \quad (6)$$

If  $k' = k$ , the integrand is 1 and is integrated over  $V$ , so the result is

$$\int \phi_{\mathbf{k},A}^{\dagger} \phi_{\mathbf{k},A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x \quad (7)$$

$$= \frac{V}{V} = 1 \quad (8)$$

If  $k' \neq k$ , consider the integral over  $x^1 = x$  (for the purposes of this derivation only,  $x$  refers to the single  $x$  dimension of the 3-vector  $\mathbf{x}$  and should not be confused with the four-vector  $x$  used in 1):

$$\int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},A} dx = \frac{1}{V} e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(k_y - k'_y)y} e^{-i(k_z - k'_z)z} \int e^{-i(k_x - k'_x)x} dx \quad (9)$$

$$= -\frac{1}{i(k_x - k'_x)V} e^{i(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})t} e^{-i(k_y - k'_y)y} e^{-i(k_z - k'_z)z} \left[ e^{-i(k_x - k'_x)x} \right]_{x=x_0}^{x=x_1} \quad (10)$$

$$= 0 \quad (11)$$

where  $x_0$  and  $x_1$  are the  $x$  limits of  $V$ , where by assumption the wave amplitude, and therefore the term  $e^{-i(k_x - k'_x)x}$ , are zero. Therefore

$$\int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},A} d^3x = \delta_{\mathbf{k},\mathbf{k}'} \quad (12)$$

The same result follows for  $\phi_{\mathbf{k},B^\dagger}$  by replacing  $kx$  by  $-kx$  throughout the derivation, so

$$\int \phi_{\mathbf{k},B^\dagger}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = \delta_{\mathbf{k},\mathbf{k}'} \quad (13)$$

For mixed terms, we have

$$\int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = \frac{1}{V} \int e^{i(k'+k)x} d^3x \quad (14)$$

In this case, the exponent cannot be zero, so the integral always comes out to zero, so that

$$\int \phi_{\mathbf{k}',A}^\dagger \phi_{\mathbf{k},B^\dagger} d^3x = 0 \quad (15)$$

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