

KLEIN-GORDON EQUATION - PLANE WAVE SOLUTIONS

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The Klein-Gordon equation was one of the first attempts at producing a relativistic quantum theory. In natural units, the equation is

$$(\partial_\mu \partial^\mu + m^2) \phi = 0 \quad (1)$$

This equation also results from the Euler-Lagrange equation for a scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \quad (2)$$

This is the Lagrangian for zero potential $V(\phi) = 0$.

To write solutions to equation 1, we can introduce some new notation. In natural units, the four-momentum is the four-vector

$$p_\mu = [E, p_i] \quad (3)$$

$$= [E, -p^i] \quad (4)$$

The scalar product of four-momentum with a spacetime vector x^μ is therefore

$$px \equiv p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x} \quad (5)$$

For a plane wave with angular frequency ω , Planck's relation is $E = \hbar\omega = \omega$, and the wave vector \mathbf{k} has components in the three spatial directions of $2\pi/\lambda_i$, where λ_i is the component of the wavelength in direction x_i . For example, a wave moving in the x_1 direction has $\lambda_2 = \lambda_3 = \infty$, so $\mathbf{k} = [k_1, 0, 0]$. The four-vector k^μ is

$$k^\mu = [\omega, \mathbf{k}] \quad (6)$$

and since $\mathbf{p} = \mathbf{k}$ in natural units, we have

$$kx = k_\mu x^\mu = p_\mu x^\mu = px \quad (7)$$

A plane wave solution to 1 turns out to be

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \quad (8)$$

We can see this by direct substitution. Consider one term $\phi_{\mathbf{k}}$ from the sum. Then

$$\phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-ikx} + B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \quad (9)$$

$$\partial^{\mu} \phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-ik_{\mu} A_{\mathbf{k}} e^{-ikx} + ik_{\mu} B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \quad (10)$$

$$\partial_{\mu} \partial^{\mu} \phi_{\mathbf{k}} = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-k_{\mu} k^{\mu} A_{\mathbf{k}} e^{-ikx} - k_{\mu} k^{\mu} B_{\mathbf{k}}^{\dagger} e^{ikx} \right) \quad (11)$$

However, using the invariant scalar $p_{\mu} p^{\mu}$ from relativity:

$$k_{\mu} k^{\mu} = p_{\mu} p^{\mu} = E^2 - p^2 = m^2 \quad (12)$$

Thus

$$\partial_{\mu} \partial^{\mu} \phi_{\mathbf{k}} = -m^2 \phi_{\mathbf{k}} \quad (13)$$

so 1 is true for a single component $\phi_{\mathbf{k}}$. Since the solution 8 is a linear combination of such solutions, and the original differential equation is linear, then 8 is also a solution. [The normalization factor $1/\sqrt{2V\omega_{\mathbf{k}}}$ is irrelevant in proving that 8 is a solution; it's just there to make future calculations easier.]

Note that the first term (involving $A_{\mathbf{k}}$) is also a solution of the free-particle Schrödinger equation, which is

$$i \frac{\partial \phi_S}{\partial t} = -\frac{1}{2m} \nabla^2 \phi_S \quad (14)$$

If we take

$$\phi_S = \sum_{\mathbf{k}} A_{\mathbf{k}} e^{-ikx} \quad (15)$$

then

$$i \frac{\partial \phi_S}{\partial t} = \sum_{\mathbf{k}} E_{\mathbf{k}} A_{\mathbf{k}} e^{-ikx} \quad (16)$$

$$-\frac{1}{2m} \nabla^2 \phi_S = \sum_{\mathbf{k}} \frac{k^2}{2m} A_{\mathbf{k}} e^{-ikx} \quad (17)$$

For a free particle, the energy $E_{\mathbf{k}} = \frac{p^2}{2m} = \frac{k^2}{2m}$, so the Schrödinger equation is satisfied. The second term (with $B_{\mathbf{k}}^\dagger$) does *not* satisfy the Schrödinger equation, since in that case we get

$$\phi_S = \sum_{\mathbf{k}} B_{\mathbf{k}}^\dagger e^{ikx} \quad (18)$$

$$i \frac{\partial \phi_S}{\partial t} = - \sum_{\mathbf{k}} E_{\mathbf{k}} B_{\mathbf{k}}^\dagger e^{ikx} \quad (19)$$

$$-\frac{1}{2m} \nabla^2 \phi_S = \sum_{\mathbf{k}} \frac{k^2}{2m} B_{\mathbf{k}}^\dagger e^{ikx} \quad (20)$$

The extra minus sign means the two sides don't match.

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