KLEIN-GORDON EQUATION - PROBABILITY DENSITY AND CURRENT

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Post date: 11 August 2023.

In non-relativistic quantum mechanics governed by the Schrödinger equation, the probability density is given by

$$\rho = \Psi^{\dagger}\Psi \tag{1}$$

and the probability current is given by (generalizing our earlier result to 3-d and using natural units):

$$\mathbf{J} = \frac{i}{2m}\left(\Psi\nabla\Psi^{\dagger} - \Psi^{\dagger}\nabla\Psi\right) \tag{2}$$

The continuity equation for probability is then

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{3}$$

We'll now look at how these results appear in relativistic quantum mechanics, using the Klein-Gordon equation:

$$\frac{\partial^2\phi}{\partial t^2} = \left(\nabla^2 - \mu^2\right)\phi = 0 \tag{4}$$

We can multiply this equation by $\phi^{\dagger}$ and then subtract the hermitian conjugate of the result from the original equation to get

$$\phi^{\dagger}\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^{\dagger}}{\partial t^2} = \phi^{\dagger}\left(\nabla^2 - \mu^2\right)\phi - \phi\left(\nabla^2 - \mu^2\right)\phi^{\dagger} \tag{5}$$

$$= \phi^{\dagger}\nabla^2\phi - \phi\nabla^2\phi^{\dagger} \tag{6}$$

The LHS can be written as

$$\phi^{\dagger}\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^{\dagger}}{\partial t^2} = \frac{\partial}{\partial t}\left(\phi^{\dagger}\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^{\dagger}}{\partial t}\right) \tag{7}$$

(use the product rule on the RHS and cancel terms).

The RHS of 6 can be written as (use the product rule again):

$$\phi^\dagger\nabla^2\phi - \phi\nabla^2\phi^\dagger = \nabla\cdot\left(\phi^\dagger\nabla\phi - \phi\nabla\phi^\dagger\right) \tag{8}$$

We can write this as a continuity equation for the Klein-Gordon equation, with the following definitions:

$$\rho \equiv i\left(\phi^\dagger\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^\dagger}{\partial t}\right) \tag{9}$$

$$\mathbf{j} = -i\left(\phi^\dagger\nabla\phi - \phi\nabla\phi^\dagger\right) \tag{10}$$

[The extra i is introduced to make ρ and $\mathbf{j}$ real. Note that the factor within the parentheses in both expressions is a complex quantity minus its complex conjugate, which always gives a pure imaginary term. Thus multiplying by i ensures the result is real.]

Then

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{11}$$

We can put this in 4-vector form if we use (for some 3-vector $\mathbf{A}$):

$$\nabla\cdot\mathbf{A} = -\partial^i a_i \tag{12}$$

where the implied sum over i is from $i = 1$ to $i = 3$ (spatial coordinates), and the minus sign appears because we've raised the index on ∂^i. If we define

$$j_i = i\left(\phi^\dagger\partial_i\phi - \phi\partial_i\phi^\dagger\right) \tag{13}$$

(that is, the negative of 10), then $\nabla\cdot\mathbf{j} = \partial^i j_i$. To make j_μ into a 4-vector, we add $j_0 = \rho$ and we get

$$\frac{\partial j_0}{\partial t} + \partial^i j_i = \partial^\mu j_\mu = 0 \tag{14}$$

[Note that my definition of j_i is the negative of the middle term in Klauber's equation 3-21, although raising the i index agrees with the last term in 3-21. I can't see how his middle and last equations for j_i and j^i can both be right, since raising the i in the middle equation for j_i merely raises the $\phi_{,i}$ to $\phi^{,i}$ without changing the sign.]

The curious thing about the Klein-Gordon equation is that its probability density ρ in 9 need not be positive, depending on the values of ϕ and its time derivative. To see how this can affect the physical meaning of the equation, consider the general plane wave solution to the Klein-Gordon equation

$$\phi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(A_{\mathbf{k}}e^{-ikx}+B_{\mathbf{k}}^{\dagger}e^{ikx}\right) \tag{15}$$

Klauber explores this starting with his equation 3-24, where he takes a test solution in which all $B_{\mathbf{k}}^{\dagger}=0$ and shows that $\int\rho\,d^3x=\sum_{\mathbf{k}}|A_{\mathbf{k}}|^2=1$ so that in this case, the total probability of finding the system in some state is +1 as it should be. Let's see what happens if we take all $A_{\mathbf{k}}=0$. In that case, 9 becomes

$$\begin{aligned}\rho=&\,i\left[\sum_{\mathbf{k}}\frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\left[\sum_{\mathbf{k}'}\frac{i\omega_{\mathbf{k}'}B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]-\\&i\left[\sum_{\mathbf{k}'}\frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]\left[\sum_{\mathbf{k}}\frac{-i\omega_{\mathbf{k}}B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\end{aligned} \tag{16}$$

$$\begin{aligned}=&-\left[\sum_{\mathbf{k}}\frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\left[\sum_{\mathbf{k}'}\frac{\omega_{\mathbf{k}'}B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]-\\&\left[\sum_{\mathbf{k}'}\frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]\left[\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\end{aligned} \tag{17}$$

We now wish to calculate $\int\rho\,d^3x$. We can use the orthonormality of solutions to do the integral. We have

$$\frac{1}{V}\int e^{i(k'-k)x}d^3x=\delta_{\mathbf{k},\mathbf{k}'} \tag{18}$$

We get

$$-\int\left[\sum_{\mathbf{k}}\frac{B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]\left[\sum_{\mathbf{k}'}\frac{\omega_{\mathbf{k}'}B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]d^3x=-\sum_{\mathbf{k}}\frac{|B_{\mathbf{k}}|^2}{2} \tag{19}$$

$$-\int\left[\sum_{\mathbf{k}'}\frac{B_{\mathbf{k}'}^{\dagger}}{\sqrt{2\omega_{\mathbf{k}'}V}}e^{ik'x}\right]\left[\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}B_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}V}}e^{-ikx}\right]d^3x=-\sum_{\mathbf{k}}\frac{|B_{\mathbf{k}}|^2}{2} \tag{20}$$

$$\int\rho\,d^3x=-\sum_{\mathbf{k}}|B_{\mathbf{k}}|^2 \tag{21}$$

Thus the total probability of finding the system in one of the states **k** is negative.

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