

LAGRANGIAN TO EQUATION OF MOTION - THREE EXAMPLES

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Here are three more simple examples of Lagrangians and their associated equations of motion. The first two examples are Lagrangians with interaction terms; the third example is for the free field Klein-Gordon system.

Example 1. For a massive scalar field coupled to a source $J(x)$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} [\partial_\mu \phi(x)]^2 - \frac{1}{2} m^2 [\phi(x)]^2 + J(x) \phi(x) \quad (1)$$

Using the Euler-Lagrange equations to get the equation of motion, we have

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (2)$$

So we have

$$-\frac{\partial \mathcal{L}}{\partial \phi} = m^2 \phi(x) - J(x) \quad (3)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi \quad (4)$$

The equation of motion is therefore

$$(\partial_\mu \partial^\mu \phi + m^2) \phi(x) = J(x) \quad (5)$$

Example 2. A possible Lagrangian for two interacting scalar fields is given by L&B's equation 7.14:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1)^2 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 - \frac{1}{2} m^2 \phi_2^2 - g (\phi_1^2 + \phi_2^2)^2 \quad (6)$$

The interaction comes from the last term, which contains terms that aren't quadratic in the fields.

Applying the Euler-Lagrange equations, we have (for $i = 1, 2$)

$$-\frac{\partial \mathcal{L}}{\partial \phi_i} = m^2 \phi_i + 2g (\phi_1^2 + \phi_2^2) \times 2\phi_i \quad (7)$$

$$= m^2 \phi_i + 4g (\phi_1^2 + \phi_2^2) \phi_i \quad (8)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} = \partial^\mu \partial_\mu \phi_i \quad (9)$$

We therefore have two equations of motion, one for each ϕ_i :

$$\partial^\mu \partial_\mu \phi_1 + m^2 \phi_1 + 4g (\phi_1^2 + \phi_2^2) \phi_1 = 0 \quad (10)$$

$$\partial^\mu \partial_\mu \phi_2 + m^2 \phi_2 + 4g (\phi_1^2 + \phi_2^2) \phi_2 = 0 \quad (11)$$

Example 3. The Klein-Gordon Lagrangian is given by L&B's equation 7.7:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \quad (12)$$

This can be written as

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (13)$$

Using L&B's metric

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (14)$$

Raising a spatial index introduces a minus sign, while raising the time index doesn't change anything, so

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) \quad (15)$$

Therefore, 13 is equivalent to

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \nabla \phi \cdot \nabla \phi - \frac{1}{2} m^2 \phi^2 \quad (16)$$

$$= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \quad (17)$$

The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \quad (18)$$

The Hamiltonian density is

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} \quad (19)$$

$$= \dot{\phi}^2 - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \quad (20)$$

$$= \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \quad (21)$$

The generalized momentum density is, using 13

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \quad (22)$$

$$= \partial^\mu \phi \quad (23)$$

The time component is

$$\Pi^0 = \dot{\phi} = \pi \quad (24)$$