

## LAGRANGIANS AND THE PRINCIPLE OF LEAST ACTION

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A fundamental application of functional derivatives is in the derivation of the principle of least action and the Euler-Lagrange equation. A particle with mass  $m$  following some trajectory  $x(t)$  will have a potential energy  $V(x)$  and a kinetic energy  $T(x)$  at each point on the trajectory. If the particle follows the trajectory between times  $t = 0$  and  $t = \tau$ , then these energies have averages given as functionals

$$\bar{V}[x] = \frac{1}{\tau} \int_0^\tau V(x(t)) dt \quad (1)$$

$$\bar{T}[x] = \frac{1}{\tau} \int_0^\tau T(x(t)) dt \quad (2)$$

$$= \frac{m}{2\tau} \int_0^\tau \dot{x}^2 dt \quad (3)$$

From example 1 in the earlier post, we have

$$\frac{\delta \bar{V}[x]}{\delta x(t)} = \frac{1}{\tau} \frac{\partial V}{\partial x} \quad (4)$$

From example 2 in the same post, we have

$$\frac{\delta \bar{T}[x]}{\delta x(t)} = \frac{1}{\tau} \left( \frac{\partial T}{\partial x} - \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}} \right] \right) \quad (5)$$

$$= \frac{1}{\tau} (0 - m\dot{x}) \quad (6)$$

$$= -\frac{m\ddot{x}}{\tau} \quad (7)$$

In classical physics, all forces are ultimately represented by conservative forces (gravity and electromagnetism), so a force can be represented by the gradient of a potential:  $F = -\frac{\partial V}{\partial x}$ . Newton's law  $F = ma$  then says

$$m\ddot{x} = -\frac{\partial V}{\partial x} \quad (8)$$

so in terms of functional derivatives

$$\frac{\delta \bar{T}[x]}{\delta x(t)} = \frac{\delta \bar{V}[x]}{\delta x(t)} \quad (9)$$

That is, the variation in the average kinetic energy over the path is equal to the variation in the average potential energy. This suggests that the quantity

$$L \equiv T - V \quad (10)$$

known as the *Lagrangian*, is in some sense fundamental, since the variation in its integral over a path is zero. To give a name to this integral, we define the *action*  $S$  as

$$S \equiv \int_0^\tau L dt \quad (11)$$

giving rise to the condition

$$\frac{\delta S[x]}{\delta x(t)} = 0 \quad (12)$$

Technically, this means that the action is stationary when the trajectory  $x(t)$  is the actual trajectory followed by a particle. In practice, the action nearly always turns out to be a minimum, so this is known as the *principle of least action*. Varying the trajectory slightly causes the action to increase and gives a path that is not followed by the particle.

If we know the Lagrangian for a particle (or more generally, for a system of particles), we can work out the equation(s) of motion by applying the principle of least action. To do this, we consider the Lagrangian as a function of a generalized coordinate  $x(t)$  and its first derivative  $\dot{x}(t)$ :

$$L = L(x, \dot{x}) \quad (13)$$

We can then apply the principle of least action:

$$\frac{\delta S[x]}{\delta x(t)} = \frac{\delta}{\delta x(t)} \left[ \int_0^\tau L(x, \dot{x}) dt \right] = 0 \quad (14)$$

This has the same form as example 2 from the earlier post that we used above, so we get

$$\frac{\delta}{\delta x(t)} \left[ \int_0^\tau L(x, \dot{x}) dt \right] = \frac{\partial L}{\partial x} - \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{x}} \right] = 0 \quad (15)$$

Thus we get the *Euler-Lagrange equation* for a single particle moving in one dimension

$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0} \quad (16)$$

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**Example.** As a simple example, suppose we have a particle whose potential and kinetic energies are given by

$$V(x, \dot{x}) = \frac{1}{2}kx^2 \quad (17)$$

$$T(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 \quad (18)$$

Then the equation of motion is given by

$$L(x, \dot{x}) = \frac{1}{2}(m\dot{x}^2 - kx^2) \quad (19)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -kx - m\ddot{x} = 0 \quad (20)$$

$$m\ddot{x} = -kx \quad (21)$$

This is the equation of motion for a mass on a spring with spring constant  $k$ .

#### PINGBACKS

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