

LAGRANGIANS FOR ELASTIC MEDIA

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As another application of the principle of least action we will look at a vibrating string. Suppose we have a string stretched between two points so that its tension is \mathcal{T} . If we pluck the string so that it starts vibrating, then at a point x on the line connecting the points to which the ends of the string are joined, the displacement from equilibrium of a point on the string is given by $\psi(x, t)$. Assuming that a point on the string can move only up and down (that is, perpendicular to the line joining the ends), then the kinetic energy of a segment of length dx is (given that the string's linear density is ρ):

$$dT = \frac{1}{2} \rho dx \dot{\psi}^2 \quad (1)$$

To get the potential energy, we can use the following argument. If a segment of length dx is displaced from equilibrium by a distance $d\psi$ by the action of the tension \mathcal{T} , then the stretched length of that segment is

$$ds = \sqrt{dx^2 + d\psi^2} \approx dx \left(1 + \frac{1}{2} \left(\frac{d\psi}{dx} \right)^2 \right) \quad (2)$$

to first order in dx . The work done to stretch the string is the force \mathcal{T} times the distance stretched, which is $ds - dx$, so the work done, which is equal to the potential energy, is

$$dV = \mathcal{T} (ds - dx) = \frac{\mathcal{T}}{2} \left(\frac{d\psi}{dx} \right)^2 dx \quad (3)$$

We can now introduce the *Lagrangian density* (effectively, the Lagrangian per unit length):

$$\mathcal{L} \equiv \frac{dL}{dx} = \frac{dT - dV}{dx} = \frac{\rho}{2} (\partial_t \psi)^2 - \frac{\mathcal{T}}{2} (\partial_x \psi)^2 \quad (4)$$

The action can now be written as

$$S = \int_1 \mathcal{L} dx dt \quad (5)$$

To minimize the action we now have to consider a functional that depends on a function of two independent variables. That is, we have

$$\mathcal{L} = \mathcal{L}(\psi, \partial_x \psi, \partial_t \psi) \quad (6)$$

where

$$\psi = \psi(x, t) \quad (7)$$

We can generalize the original definition of a functional derivative by considering a variation of the function ψ at one point (x_0, t_0) , that is

$$\psi(x, t) \rightarrow \psi(x, t) + \epsilon \delta(x - x_0) \delta(t - t_0) \quad (8)$$

To see how this works, consider the slightly simpler case where the function \mathcal{L} depends only on ψ and not on its derivatives. Then

$$\frac{\delta S[\psi(x, t)]}{\delta \psi(x_0, t_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int dt \int dx [\mathcal{L}(\psi + \epsilon \delta(x - x_0) \delta(t - t_0)) - \mathcal{L}(\psi)] \quad (9)$$

$$= \int dt \int dx \frac{\partial \mathcal{L}}{\partial \psi(x, t)} \delta(x - x_0) \delta(t - t_0) \quad (10)$$

$$= \int dt \frac{\partial \mathcal{L}}{\partial \psi(x_0, t)} \delta(t - t_0) \quad (11)$$

$$= \frac{\partial \mathcal{L}}{\partial \psi(x_0, t_0)} \quad (12)$$

where we used a Taylor expansion of the first line up to first order to get the second line.

To extend this to the more general case where $\mathcal{L} = \mathcal{L}(\psi, \partial_x \psi, \partial_t \psi)$ we can follow example 2 in this earlier post to get

$$\frac{\delta S[\psi(x, t)]}{\delta \psi(x_0, t_0)} = \frac{\partial \mathcal{L}}{\partial \psi(x_0, t_0)} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial (\partial_x \psi(x_0, t_0))} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\partial_t \psi(x_0, t_0))} = 0 \quad (13)$$

Applying this to 4 we get

$$\frac{\partial \mathcal{L}}{\partial \psi(x_0, t_0)} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial (\partial_x \psi(x_0, t_0))} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\partial_t \psi(x_0, t_0))} = 0 + \mathcal{T} \frac{d}{dx} (\partial_x \psi) - \rho \frac{d}{dt} (\partial_t \psi) = 0 \quad (14)$$

$$\partial_x^2 \psi = \frac{\rho}{\mathcal{T}} \partial_t^2 \psi \quad (15)$$

The final equation is the wave equation that we derived earlier when studying electromagnetic waves.

We can generalize this to a three-dimensional elastic medium by taking $\psi(x, y, z, t) = \psi(\mathbf{r}, t)$ to be a three-dimensional scalar field (actually, it's not clear exactly what ψ represents in this case; in the 'real' theory of 3-d elasticity, the displacement of an element of the elastic medium is a vector field, not a scalar field, as you would expect). We can generalize the idea of a Lagrangian density to three dimensions, so we have

$$\mathcal{L} = \frac{\rho}{2} (\partial_t \psi)^2 - \frac{\mathcal{T}}{2} (\nabla \psi)^2 \quad (16)$$

The principle of least action is now

$$\frac{\delta S[\psi(\mathbf{r}, t)]}{\delta \psi(\mathbf{r}_0, t_0)} = \frac{\delta}{\delta \psi(\mathbf{r}_0, t_0)} \int dt \int d^3 \mathbf{r} \left[\frac{\rho}{2} (\partial_t \psi)^2 - \frac{\mathcal{T}}{2} (\nabla \psi)^2 \right] \quad (17)$$

$$= \frac{\delta}{\delta \psi(\mathbf{r}_0, t_0)} \int dt \int d^3 \mathbf{r} \left[\frac{\rho}{2} (\partial_t \psi)^2 - \frac{\mathcal{T}}{2} \left[(\partial_x \psi)^2 + (\partial_y \psi)^2 + (\partial_z \psi)^2 \right] \right] \quad (18)$$

ψ is now a function of 4 independent variables, so we can use the same argument as above to calculate the functional derivative. We perturb ψ by the amount $\epsilon \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) \delta(t - t_0)$ and follow through the derivation in the same way. All the terms in the integral are of the form we looked at in example 4 in this post, so we get

$$\frac{\delta S[\psi(\mathbf{r}, t)]}{\delta \psi(\mathbf{r}_0, t_0)} = -\rho \partial_t^2 \psi + \mathcal{T} (\partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi) = 0 \quad (19)$$

$$\nabla^2 \psi = \frac{\rho}{\mathcal{T}} \partial_t^2 \psi \quad (20)$$

We get the 3-d wave equation.

PINGBACKS

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