LORENTZ TRANSFORMATION AND TRANSLATION OF CREATION AND ANNIHILATION OPERATORS

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The creation and annihilation operators $a_{\boldsymbol{p}}^{\dagger}$ and $a_{\boldsymbol{p}}$ for quantum fields are defined in analogy to the raising and lowering operators of the harmonic oscillator. We can define relativistically normalized versions of these operators in 4-momentum space as

$$\begin{aligned} \alpha^{\dagger}(p) &\equiv (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}^{\dagger} \\ \alpha(p) &\equiv (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}} \end{aligned} \tag{1}$$

In equations 2.63 to 2.67, Coleman shows that the Lorentz transformation properties of these operators are

$$\boxed{\begin{aligned} U(\Lambda)\alpha^{\dagger}(p)U^{\dagger}(\Lambda) &= \alpha^{\dagger}(\Lambda p) \\ U(\Lambda)\alpha(p)U^{\dagger}(\Lambda) &= \alpha(\Lambda p) \end{aligned}} \tag{2}$$

That is, if we apply the unitary Lorentz transformation $U(\Lambda)$ as shown to $\alpha^{\dagger}(p)$ or $\alpha(p)$, we obtain the corresponding creation or annihilation operator for a particle with the Lorentz-transformed 4-momentum Λp.

Coleman just states the behaviour of $\alpha^{\dagger}(p)$ and $\alpha(p)$ under translation in 2.72. We can see where these come from as follows.

The translation operator is given by

$$U(a) = e^{iP\cdot a} \tag{3}$$

where P is the 4-momentum operator and a is a 4-vector in spacetime. A 3-dim version of this was derived in non-relativistic quantum mechanics. An operator transforms under translation according to Coleman's eqn 1.37:

$$\Omega(x+a) = e^{iP\cdot a}\Omega(x)e^{-iP\cdot a} \tag{4}$$

There is the possibility of confusion here since the sign of the exponent depends on whether we use the 'east coast' or 'west coast' metric, that is, whether a scalar product is given by

$$x\cdot y=-x^0y^0+x^1y^1+x^2y^2+x^3y^3 \quad \text{east coast} \tag{5}$$
$$x\cdot y=+x^0y^0-x^1y^1-x^2y^2-x^3y^3 \quad \text{west coast} \tag{6}$$

Coleman uses the west coast metric, so the translation operator is

$$U(a)=e^{iP\cdot a}=e^{iP^0a^0-i\boldsymbol{P}\cdot\boldsymbol{a}} \tag{7}$$

Using 3, we have the transformation of the creation operator:

$$\alpha^\dagger(p)\to e^{iP\cdot a}\alpha^\dagger(p)e^{-iP\cdot a} \tag{8}$$

To see the effect of the translation, we apply it to the vacuum state.

$$e^{iP\cdot a}\alpha^\dagger(p)e^{-iP\cdot a}|0\rangle=e^{iP\cdot a}\alpha^\dagger(p)|0\rangle \tag{9}$$
$$=e^{iP\cdot a}|p\rangle \tag{10}$$
$$=e^{ip\cdot a}|p\rangle \tag{11}$$

In the first line, we use the fact that the momentum of the vacuum state is zero, so

$$e^{-iP\cdot a}|0\rangle=e^{-i0\cdot a}|0\rangle=|0\rangle \tag{12}$$

The second line uses the fact that $\alpha^\dagger(p)|0\rangle=|p\rangle$ and the last line applies the momentum operator P to a state with momentum p. Note that we've replaced the operator P with its eigenvalue p.

Comparing 11 with the LHS of 9, we have

$$e^{iP\cdot a}\alpha^\dagger(p)e^{-iP\cdot a}|0\rangle=e^{ip\cdot a}|p\rangle \tag{13}$$
$$=e^{ip\cdot a}\alpha^\dagger(p)|0\rangle \tag{14}$$

so the transformation of the creation operator is

$$\boxed{e^{iP\cdot a}\alpha^\dagger(p)e^{-iP\cdot a}=e^{ip\cdot a}\alpha^\dagger(p)} \tag{15}$$

as given in Coleman's eqn 2.72 (except he uses x instead of a).

To transform the annihilation operator, we have

$$e^{iP\cdot a}\alpha(p)e^{-iP\cdot a}|p\rangle=e^{-ip\cdot a}e^{iP\cdot a}\alpha(p)|p\rangle \tag{16}$$
$$=e^{-ip\cdot a}e^{iP\cdot a}|0\rangle \tag{17}$$
$$=e^{-ip\cdot a}|0\rangle \tag{18}$$
$$=e^{-ip\cdot a}\alpha(p)|p\rangle \tag{19}$$

Here we start by applying the transformed operator to a state $|p\rangle$ and use the fact that $\alpha(p)$ annihilates this state, converting it to the vacuum state $|0\rangle$. Comparing the LHS with the final result, we get the transformation for the annihilation operator:

$$\boxed{e^{iP\cdot a}\alpha(p)\,e^{-iP\cdot a}=e^{-ip\cdot a}\alpha(p)} \tag{20}$$

which agrees with Coleman's eqn 2.72. [This result could also be obtained by just taking the adjoint of 15.]

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