LORENTZ TRANSFORMATION AND TRANSLATION OF EXPLICIT QUANTUM FIELD

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Post date: 2 August 2021.

In Coleman's section 3.3, he derives the explicit form of a scalar quantum field that satisfies the 5 conditions given earlier. As his derivation is quite complete, I'll just summarize it here and fill in a few gaps.

We begin with the condition that the quantum field be a linear combination of the creation and annihilation operators, so its general form is

$$\phi(x)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[F_{\boldsymbol{p}}(x)\,\alpha(p)+G_{\boldsymbol{p}}(x)\,\alpha^{\dagger}(p)\right] \tag{1}$$

We've dropped the superscript a labelling the field in the original notation of ϕ^a since we're considering only one field. The quantities $F_{\boldsymbol{p}}(x)$ and $G_{\boldsymbol{p}}(x)$ are functions of the spacetime position x and momentum $\boldsymbol{p}$ (and by extension, the energy $\omega_{\boldsymbol{p}}$). They are just numerical functions, not operators.

Coleman considers 1 at $x=0$ so we have

$$\phi(0)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f_p\alpha(p)+g_p\alpha^{\dagger}(p)\right] \tag{2}$$

where

$$\begin{aligned} f_p &= F_{\boldsymbol{p}}(0)\\ g_p &= G_{\boldsymbol{p}}(0) \end{aligned} \tag{3}$$

We apply a Lorentz transformation, using the conditions

$$\begin{aligned} U(\Lambda)\,\alpha^{\dagger}(p)\,U^{\dagger}(\Lambda) &= \alpha^{\dagger}(\Lambda p)\\ U(\Lambda)\,\alpha(p)\,U^{\dagger}(\Lambda) &= \alpha(\Lambda p) \end{aligned} \tag{4}$$

That is, we consider the quantity

$$U(\Lambda)\,\phi(0)\,U^{\dagger}(\Lambda)=U(\Lambda)\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[f_p\alpha(p)+g_p\alpha^{\dagger}(p)\right]U^{\dagger}(\Lambda) \tag{5}$$

Coleman observes that the unitary operator $U(\Lambda)$ has no effect on the functions f_p and g_p or, in his memorable phrase, $U(\Lambda)$ goes through f_p and g_p'like beet through a baby'. Having never tried to feed beet to a baby, I can't vouch for the accuracy of this comparison. His point is that we can apply $U(\Lambda)$ directly to $\alpha(p)$ and $\alpha^{\dagger}(p)$and use 4.

Having done this, we change the variable of integration to $p' = \Lambda p$ and we end up with his eqn 3.27:

$$\phi(0) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \left[f_{\Lambda^{-1}p}\alpha(p) + g_{\Lambda^{-1}p}\alpha^{\dagger}(p) \right] \tag{6}$$

The point is that this expression for $\phi(0)$ must be the same as the original 2. The only difference between these two forms is that f_p and g_p have been replaced by $f_{\Lambda^{-1}p}$and $g_{\Lambda^{-1}p}$. Since the $\alpha(p)$s and $\alpha^{\dagger}(p)$s are independent operators, we must have

$$\begin{aligned} f_p &= f_{\Lambda^{-1}p} \\ g_p &= g_{\Lambda^{-1}p} \end{aligned} \tag{7}$$

for *all* Lorentz transformations Λ and all momenta $\boldsymbol{p}$. The only way this can occur is if $f_p = f$ and $g_p = g$ and both f and g are constants. We thus obtain the form for $\phi(0)$:

$$\phi(0) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \left[f\alpha(p) + g\alpha^{\dagger}(p) \right] \tag{8}$$

To obtain $\phi(x)$ we can translate $\phi(0)$ by an amount x and use the relations:

$$\begin{aligned} e^{iP\cdot a}\alpha^{\dagger}(p)e^{-iP\cdot a} &= e^{ip\cdot a}\alpha^{\dagger}(p) \\ e^{iP\cdot a}\alpha(p)e^{-iP\cdot a} &= e^{-ip\cdot a}\alpha(p) \end{aligned} \tag{9}$$

This gives us

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \left[fe^{-ip\cdot x}\alpha(p) + ge^{ip\cdot x}\alpha^{\dagger}(p) \right] \tag{10}$$

Comparing this with 1, we see that

$$\begin{aligned} F_{\boldsymbol{p}}(x) &= fe^{-ip\cdot x} \\ G_{\boldsymbol{p}}(x) &= ge^{ip\cdot x} \end{aligned} \tag{11}$$

so the condition that $\phi(x)$ is a linear combination of the creation and annihilation operators is satisfied.

We can verify that 10 satisfies the condition on Lorentz transformations given as condition 4 in Coleman's section 3.2:

$$U(\Lambda)^{\dagger}\phi(x)U(\Lambda)=\phi\left(\Lambda^{-1}x\right) \tag{12}$$

Because $U(\Lambda)$ is unitary, and the inverse of $U(\Lambda)$ reverses the Lorentz transformation Λ,

$$\begin{aligned} U(\Lambda)^{\dagger} &= U^{-1}(\Lambda)=U\left(\Lambda^{-1}\right) \\ U(\Lambda) &= U^{\dagger}\left(\Lambda^{-1}\right) \end{aligned} \tag{13}$$

so we can write 12 as

$$U\left(\Lambda^{-1}\right)\phi(x)U^{\dagger}\left(\Lambda^{-1}\right)=\phi\left(\Lambda^{-1}x\right) \tag{14}$$

Applying this to 10 and using 4 we have

$$U\left(\Lambda^{-1}\right)\phi(x)U^{\dagger}\left(\Lambda^{-1}\right)=U\left(\Lambda^{-1}\right)\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}\alpha(p)+ge^{ip\cdot x}\alpha^{\dagger}(p)\right]U^{\dagger}\left(\Lambda^{-1}\right) \tag{15}$$

$$\begin{aligned} &=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}U\left(\Lambda^{-1}\right)\alpha(p)U^{\dagger}\left(\Lambda^{-1}\right)+\right. \\ &\left. ge^{ip\cdot x}U\left(\Lambda^{-1}\right)\alpha^{\dagger}(p)U^{\dagger}\left(\Lambda^{-1}\right)\right] \end{aligned} \tag{16}$$

$$=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}\alpha\left(\Lambda^{-1}p\right)+ge^{ip\cdot x}\alpha^{\dagger}\left(\Lambda^{-1}p\right)\right] \tag{17}$$

$$=\phi\left(\Lambda^{-1}x\right) \tag{18}$$

This follows because the exponent $p\cdot x$ is a Lorentz scalar and the integration measure $\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}$ is a Lorentz invariant. Thus 14 is satisfied.

We can also verify Coleman's third condition, which is

$$e^{-iP\cdot y}\phi(x)e^{iP\cdot y}=\phi(x-y) \tag{19}$$

We can apply 9 with $a=-y$, so we have

$$e^{-iP\cdot y}\phi(x)e^{iP\cdot y} = e^{-iP\cdot y}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}\alpha(p)+ge^{ip\cdot x}\alpha^{\dagger}(p)\right]e^{iP\cdot y} \tag{20}$$

$$= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}e^{-iP\cdot y}\alpha(p)e^{iP\cdot y}+ge^{ip\cdot x}e^{-iP\cdot y}\alpha^{\dagger}(p)e^{iP\cdot y}\right] \tag{21}$$

$$= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot x}e^{ip\cdot y}\alpha(p)+ge^{ip\cdot x}e^{-ip\cdot y}\alpha^{\dagger}(p)\right] \tag{22}$$

$$= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\left[fe^{-ip\cdot(x-y)}\alpha(p)+ge^{ip\cdot(x-y)}\alpha^{\dagger}(p)\right] \tag{23}$$

$$= \phi(x-y) \tag{24}$$

This verifies 19.

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