LORENTZ TRANSFORMATIONS OF RELATIVISTIC STATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 29 July 2021.

The relativistic state of a single particle with 4-momentum p is defined as

$$|p\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}\,|\boldsymbol{p}\rangle \tag{1}$$

That is, it's defined in terms of the state $|\boldsymbol{p}\rangle$ with 3-momentum $\boldsymbol{p}$ and energy

$$\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + \mu^2} \tag{2}$$

We also showed earlier that integration over the Lorentz-invariant integration measure d^4p is equivalent to integration over the 3-momentum space with measure $d^3\boldsymbol{p}/2\omega_{\boldsymbol{p}}$. That is

$$\int d^4p\; f(p) = \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} f(\boldsymbol{p}) \tag{3}$$

where $p^0 = \omega_{\boldsymbol{p}}$, so that any function of the 4-momentum p becomes a function of the 3-momentum $\boldsymbol{p}$.

The unit operator can then be written as

$$1 = \frac{1}{(2\pi)^3}\int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\;|p\rangle\langle p| \tag{4}$$

Note that the states in this integral are 4-momentum states $|p\rangle$ and not 3-momentum states $|\boldsymbol{p}\rangle$.

For a Lorentz transformation Λ, we can define a unitary operator $U(\Lambda)$ which acts on a single particle ket as follows.

$$U(\Lambda)|p\rangle = |\Lambda p\rangle \tag{5}$$

This operator satisfies four properties which can be derived using methods similar to Coleman's equations 1.48 - 1.50, where he derives similar rules for the rotation operator.

First, we have

$$U(\Lambda)U^{\dagger}(\Lambda) = 1 \tag{6}$$

We can prove this as follows.

$$U(\Lambda)U^{\dagger}(\Lambda)=U(\Lambda)\frac{1}{(2\pi)^{3}}\int\frac{d^{3}\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\;|p\rangle\langle p|U^{\dagger}(\Lambda) \tag{7}$$

$$=\frac{1}{(2\pi)^{3}}\int\frac{d^{3}\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}U(\Lambda)|p\rangle\langle p|U^{\dagger}(\Lambda) \tag{8}$$

$$=\frac{1}{(2\pi)^{3}}\int\frac{d^{3}\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}|\Lambda p\rangle\langle\Lambda p| \tag{9}$$

As we showed earlier, transforming 4-momenta with a Lorentz transformation has a Jacobian of 1, so the integration measure $d^{4}p$ is unchanged. Since integrating over this measure is equivalent to integrating over 3-momenta via 3, we can define $p'\equiv\Lambda p$ and then we have

$$\frac{d^{3}\boldsymbol{p}'}{2\omega_{\boldsymbol{p}'}}=\frac{d^{3}\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \tag{10}$$

We therefore have

$$U(\Lambda)U^{\dagger}(\Lambda)=\frac{1}{(2\pi)^{3}}\int\frac{d^{3}\boldsymbol{p}'}{2\omega_{\boldsymbol{p}'}}\left|p'\right\rangle\left\langle p'\right|=1 \tag{11}$$

The next property is

$$U(1)=1 \tag{12}$$

which follows directly from 5, since if $\Lambda=1$, it is the identity matrix which amounts to no transformation at all, which leaves the state unchanged.

Next, we have the property

$$U(\Lambda_{1})U(\Lambda_{2})=U(\Lambda_{1}\Lambda_{2}) \tag{13}$$

We can see this directly from 5:

$$U(\Lambda_{1})U(\Lambda_{2})|p\rangle=U(\Lambda_{1})|\Lambda_{2}p\rangle \tag{14}$$

$$=|\Lambda_{1}\Lambda_{2}p\rangle \tag{15}$$

$$=U(\Lambda_{1}\Lambda_{2})|p\rangle \tag{16}$$

Finally we have

$$U^{\dagger}(\Lambda)PU(\Lambda)=\Lambda P \tag{17}$$

where P is the 4-momentum operator.

Because U is unitary, we have $U^{\dagger}=U^{-1}$ (we proved this when we proved 6) so

$$U^{\dagger}(\Lambda)\, P U(\Lambda) = U^{-1}(\Lambda)\, P \left(U^{\dagger}(\Lambda)\right)^{-1} \tag{18}$$

$$= U\left(\Lambda^{-1}\right) P U^{\dagger}\left(\Lambda^{-1}\right) \tag{19}$$

This follows because

$$U^{-1}(\Lambda)\, U(\Lambda) = 1 \tag{20}$$

and

$$U\left(\Lambda^{-1}\right) U(\Lambda) = U\left(\Lambda^{-1}\Lambda\right) = U(1) = 1 \tag{21}$$

Therefore

$$U\left(\Lambda^{-1}\right) = U^{-1}(\Lambda) \tag{22}$$

Continuing from 19 and using 4

$$UU^{\dagger}(\Lambda)\, P U(\Lambda) = U\left(\Lambda^{-1}\right) P \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\ |p\rangle \langle p| U^{\dagger}\left(\Lambda^{-1}\right) \tag{23}$$

$$= \frac{1}{(2\pi)^3} U\left(\Lambda^{-1}\right) \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\ p\, |p\rangle \langle p| U^{\dagger}\left(\Lambda^{-1}\right) \tag{24}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\ p U\left(\Lambda^{-1}\right) |p\rangle \langle p| U^{\dagger}\left(\Lambda^{-1}\right) \tag{25}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\ p \left|\Lambda^{-1}p\right\rangle \left\langle\Lambda^{-1}p\right| \tag{26}$$

We now let $p = \Lambda p'$ and use the invariance of the integration measure $\frac{d^3\boldsymbol{p}'}{2\omega_{\boldsymbol{p}'}} = \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}$:

$$U^{\dagger}(\Lambda)\, P U(\Lambda) = \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{2\omega_{\boldsymbol{p}'}} \Lambda p' \left|p'\right\rangle \left\langle p'\right| \tag{27}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \Lambda p\, |p\rangle \langle p| \tag{28}$$

$$= \Lambda P \left[\frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}\ |p\rangle \langle p|\right] \tag{29}$$

$$= \Lambda P \tag{30}$$

where in the third line, the integral in the brackets is the unit operator from 4. QED.