LORENZ GAUGE IN 4D SPACETIME FORM

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When solving Maxwell's equations, the important thing is finding the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, since these are quantities that can be measured. We can write Maxwell's equations in terms of potentials, but potentials are merely a mathematical fiction that allows for easier computation, and do not represent observable quantities.

We've seen that, in 3d form, the potentials can be modified without affecting the solutions to Maxwell's equations. Such modifications result in different gauges. Any transformation of the form

$$\begin{aligned} \boldsymbol{A}' &= \boldsymbol{A} - \nabla f \\ \Phi' &= \Phi + \frac{\partial f}{\partial t} \end{aligned} \tag{1}$$

where $f = f(\boldsymbol{x}, t)$ is an arbitrary scalar field, will also solve Maxwell's equations if $\boldsymbol{A}$ and Φ do. We can therefore try to find a function f that makes the solutions easier to find.

We can write 1 in 4d notation as

$$A'^{\mu} = A^{\mu} - \partial_{\mu} f \tag{2}$$

$$= A^{\mu} + \partial^{\mu} f \tag{3}$$

where the sign change comes from raising the index of the spatial derivatives.

Maxwell's equations in 4d formare, for free fields

$$\partial^{\alpha}\partial_{\alpha}A^{\mu} - \partial^{\mu}(\partial_{\nu}A^{\nu}) = 0 \tag{4}$$

As it stands, these equations can be difficult to solve. We can, however, choose a gauge such that the second term is zero. This is the Lorenz (not Lorentz!) gauge. We've met the Lorenz gauge in 3d notation before where it has the form

$$\nabla^2\Phi-\frac{\partial^2\Phi}{\partial t^2}=0$$
$$\nabla^2\boldsymbol{A}-\frac{\partial^2\boldsymbol{A}}{\partial t^2}=0 \tag{5}$$

In 4d form, we would like $\partial_\nu A^\nu=0$ in 4. Suppose we have a solution A^μ and change the gauge according to 3, giving A'^μ. We now calculate

$$\partial_\nu A'^\nu=\partial_\nu A^\nu+\partial_\nu\partial^\nu f \tag{6}$$

If we want $\partial_\nu A'^\nu=0$, then we would require

$$\partial_\nu A^\nu=-\partial_\nu\partial^\nu f \tag{7}$$

Since we know A^ν, we could in principle solve this equation to find f. Since f in 1 can be any function we like, there is no restriction on it apart from it being a solution to 7, so it is always possible to choose a gauge such that $\partial_\nu A^\nu=0$. This is the Lorenz gauge in 4d form.

Choosing this gauge gives the simplified form of Maxwell's equations for free fields

$$\partial^\alpha\partial_\alpha A^\mu=0 \tag{8}$$

This is the wave equation, and thus the solution of Maxwell's equation in empty space gives us travelling waves with speed $c=1$.

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