

## MAXWELL'S EQUATIONS FROM THE ELECTROMAGNETIC FIELD TENSOR

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We've seen before how Maxwell's equations can be obtained from the electromagnetic field tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \quad (1)$$

The sign convention used in L&B is different from that used by Moore and Lahiri & Pal.

Just to review this, see this post where we show that the relation

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (2)$$

yields two of Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (4)$$

The other relation is

$$\partial_\mu F^{\mu\nu} = J^\nu \quad (5)$$

gives the other two Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho \quad (6)$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (7)$$

From the facts that second derivatives are symmetric and  $F_{\mu\nu}$  is antisymmetric, we have

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\mu\nu} \quad (8)$$

$$= -\partial_\nu \partial_\mu F^{\nu\mu} \quad (9)$$

$$= -\partial_\mu \partial_\nu F^{\mu\nu} \quad (10)$$

In the last line, we merely swapped the indices  $\mu$  and  $\nu$  which is permissible since they are just dummy indices that are summed. Thus  $\partial_\mu \partial_\nu F^{\mu\nu}$  is equal to its negative, so it must be zero:

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \quad (11)$$

Thus from 5 we have

$$\partial_\nu J^\nu = 0 \quad (12)$$

Since  $J^\nu$  is the four-vector

$$J^\nu = [\rho, \mathbf{J}] \quad (13)$$

and

$$\partial_\nu = \left[ \frac{\partial}{\partial t}, \nabla \right] \quad (14)$$

this gives

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \quad (15)$$

which is the usual continuity equation for charge. The rate of change of charge density at a given point is equal to the negative divergence of the current, which carries charge into or away from that point.