MAXWELL'S EQUATIONS IN 4D SPACETIME FORM

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We can combine the electric potential Φ and magnetic vector potential $\boldsymbol{A}$ into a 4-vector A^{μ} to get a 4d form of electromagnetic potential. We have

$$A^{\mu}\left(x\right)=\left[\begin{array}{c}\Phi\left(x\right)\\A^{1}\left(x\right)\\A^{2}\left(x\right)\\A^{3}\left(x\right)\end{array}\right] \tag{1}$$

where $x=\left(\boldsymbol{x},t\right)$ represents 4d spacetime coordinates.

From this, we define the electromagnetic field tensor $F^{\mu\nu}\left(x\right)$ as

$$F^{\mu\nu}\left(x\right)=\partial^{\nu}A^{\mu}-\partial^{\mu}A^{\nu} \tag{2}$$

For free fields (that is, no sources such as charge or current), Maxwell's equations are

$$\begin{aligned}\nabla\cdot\boldsymbol{E}&=0\\\nabla\cdot\boldsymbol{B}&=0\\\nabla\times\boldsymbol{E}&=-\frac{\partial\boldsymbol{B}}{\partial t}\\\nabla\times\boldsymbol{B}&=\frac{\partial\boldsymbol{E}}{\partial t}\end{aligned} \tag{3}$$

These equations use units such that $\mu_{0}\epsilon_{0}=1$. These equations can be written in terms of potentials using

$$\boldsymbol{E}=-\nabla\Phi-\frac{\partial\boldsymbol{A}}{\partial t} \tag{4}$$

$$\boldsymbol{B}=\nabla\times\boldsymbol{A} \tag{5}$$

From 2, we see that $F^{\mu\nu}$ is antisymmetric, so all its diagonal elements are zero. We can then calculate the off-diagonal elements and compare them with 4 and 5. For example

$$F^{12} = -F^{21} = \partial^2 A^1 - \partial^1 A^2 \tag{6}$$
$$= -\partial_y A^1 + \partial_x A^2 \tag{7}$$
$$= (\nabla \times \boldsymbol{A})_z \tag{8}$$
$$= B^3 \tag{9}$$

The sign change in line 2 is due to lowering the index on the derivative term, since

$$\partial^2 = \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x^2} = -\partial_2 \tag{10}$$

Similarly,

$$\begin{aligned} F^{13} &= -B^2 = -F^{31} \\ F^{23} &= B^1 = -F^{32} \end{aligned} \tag{11}$$

The component $F^{10} = -F^{01}$ is

$$F^{10} = \partial^0 A^1 - \partial^1 A^0 \tag{12}$$
$$= \partial_t A^1 + \partial_x \Phi \tag{13}$$
$$= \left[\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right]_x \tag{14}$$
$$= -E_x \tag{15}$$

where the last line comes from comparison with 4.

Inserting 4 and 5 into 3 gives us, after using a few vector calculus identities, Maxwell's equations for free fields in terms of the potentials:

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = 0 \tag{16}$$
$$\nabla^2\boldsymbol{A} - \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{\partial \Phi}{\partial t}\right) = 0 \tag{17}$$

These equations can be combined into a single 4d equation

$$\partial^\alpha \partial_\alpha A^\mu - \partial^\mu (\partial_\nu A^\nu) = 0 \tag{18}$$

We can verify this by looking at specific values for μ. For $\mu = i$ = 1, 2 or 3, we have for the first term

$$\partial^\alpha \partial_\alpha A^i = \frac{\partial^2 A^i}{\partial t^2} + \frac{\partial^2 A^i}{\partial x^k \partial x_k} \tag{19}$$
$$= \frac{\partial^2 A^i}{\partial t^2} - \frac{\partial^2 A^i}{\partial x^k \partial x^k} \tag{20}$$
$$= \frac{\partial^2 A^i}{\partial t^2} - \nabla^2 A^i \tag{21}$$
$$= -\left[\nabla^2 \boldsymbol{A} - \frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right]_i \tag{22}$$

For the second term, we have

$$-\partial^i \left(\partial_\nu A^\nu\right) = \partial_i \left(\partial_\nu A^\nu\right) \tag{23}$$
$$= \left[\nabla \left(\nabla \cdot \boldsymbol{A} + \frac{\partial \Phi}{\partial t}\right)\right]_i \tag{24}$$

Combining these terms we have 17.

For $\mu = 0$, we have from 18

$$\partial^\alpha \partial_\alpha A^0 - \partial^0 \left(\partial_\nu A^\nu\right) = 0 \tag{25}$$

The first term is

$$\partial^\alpha \partial_\alpha A^0 = \frac{\partial^2 A^0}{\partial t^2} + \frac{\partial^2 A^0}{\partial x^i \partial x_i} \tag{26}$$
$$= \frac{\partial^2 A^0}{\partial t^2} - \frac{\partial^2 A^0}{\partial x^i \partial x^i} \tag{27}$$
$$= \frac{\partial^2 A^0}{\partial t^2} - \nabla^2 A^0 \tag{28}$$

The second term is

$$-\partial^0 \left(\partial_\nu A^\nu\right) = -\frac{\partial^2 A^0}{\partial t^2} - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i} A^i\right) \tag{29}$$
$$= -\frac{\partial^2 A^0}{\partial t^2} - \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{A}\right) \tag{30}$$

Combining 28 and 30 we get

$$\partial^{\alpha}\partial_{\alpha}A^{0}-\partial^{0}\left(\partial_{\nu}A^{\nu}\right)=\frac{\partial^{2}A^{0}}{\partial t^{2}}-\nabla^{2}A^{0}-\frac{\partial^{2}A^{0}}{\partial t^{2}}-\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}\right) \tag{31}$$

$$=-\nabla^{2}A^{0}-\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}\right)=0 \tag{32}$$

Multiplying through by -1 gives us 16.

We therefore have the 4d form of Maxwell's equations for free fields

$$\boxed{\partial^{\alpha}\partial_{\alpha}A^{\mu}-\partial^{\mu}\left(\partial_{\nu}A^{\nu}\right)=0} \tag{33}$$

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