MAXWELL'S EQUATIONS WITH SOURCES IN 4D SPACETIME

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 10 November 2023.

We've seen how Maxwell's equations for electromagnetism can be written in 4d spacetime form for the case of free fields. We have, in traditional 3d form

$$\begin{aligned}
\nabla\cdot\boldsymbol{E} &= 0\\
\nabla\cdot\boldsymbol{B} &= 0\\
\nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t} \qquad (1)\\
\nabla\times\boldsymbol{B} &= \frac{\partial\boldsymbol{E}}{\partial t}
\end{aligned}$$

These equations use units such that $\mu_0\epsilon_0 = 1$. These equations can be written in terms of potentials using

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial\boldsymbol{A}}{\partial t} \qquad (2)$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \qquad (3)$$

Inserting 2 and 3 into 1 gives us, after using a few vector calculus identities, Maxwell's equations for free fields in terms of the potentials:

$$-\nabla^2\Phi - \frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}\right) = 0 \qquad (4)$$

$$\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} + \nabla\left(\nabla\cdot\boldsymbol{A} + \frac{\partial\Phi}{\partial t}\right) = 0 \qquad (5)$$

We can combine the electric potential Φ and magnetic vector potential $\boldsymbol{A}$ into a 4-vector A^μ to get a 4d form of electromagnetic potential. We have

$$A^\mu(x) = \begin{bmatrix}\Phi(x)\\ A^1(x)\\ A^2(x)\\ A^3(x)\end{bmatrix} \qquad (6)$$

where $x = (\boldsymbol{x}, t)$ represents 4d spacetime coordinates.

Using this 4d form of the potential A^{μ}, Maxwell'sequations can be combined into a single 4d equation

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}-\partial^{\mu}\left(\partial_{\nu}A^{\nu}\right)=0 \tag{7}$$

If we now introduce charges and currents, then we can write Maxwell's equation in terms of potentials as

$$-\nabla^{2}\Phi-\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}\right)=\rho \tag{8}$$

$$\frac{\partial^{2}\boldsymbol{A}}{\partial t^{2}}-\nabla^{2}\boldsymbol{A}+\nabla\left(\nabla\cdot\boldsymbol{A}+\frac{\partial\Phi}{\partial t}\right)=\boldsymbol{j} \tag{9}$$

Here, ρ is the charge density and $\boldsymbol{j}$ is the 3d current density.

These equations can be written in 4d form as 7 with a 4d source vector on the RHS:

$$\partial^{\alpha}\partial_{\alpha}A^{\mu}-\partial^{\mu}\left(\partial_{\nu}A^{\nu}\right)=-ej^{\mu} \tag{10}$$

Here, this source vector is defined so that

$$-ej^{\mu}=\left[\begin{array}{c}\rho\\ j^{1}\\ j^{2}\\ j^{3}\end{array}\right] \tag{11}$$

where the 3d current density is

$$\boldsymbol{j}=\left(j^{1},j^{2},j^{3}\right) \tag{12}$$

The only difference between 10 and 7 is the source term $-ej^{\mu}$ on the RHS of 10, so we can use the same derivation to show that the LHS of 10 gives us the LHS of 8 and 9. In particular, for $\mu=0$, we have from 10

$$\partial^{\alpha}\partial_{\alpha}\Phi-\partial_{t}\left(\partial_{\nu}A^{\nu}\right)=\rho \tag{13}$$

Expanding the sums gives

$$\frac{\partial^{2}\Phi}{\partial t^{2}}-\nabla^{2}\Phi-\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}+\nabla\cdot\boldsymbol{A}\right)=\rho \tag{14}$$

$$-\nabla^{2}\Phi-\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{A}\right)=\rho \tag{15}$$

which agrees with 8.

For $\mu=i$ (a spatial index), we have

$$\frac{\partial^2 A^i}{\partial t^2} - \nabla^2 A^i - \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = j^i \tag{16}$$

We now use

$$\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i} \tag{17}$$

and we have

$$\frac{\partial^2 A^i}{\partial t^2} - \nabla^2 A^i + \frac{\partial}{\partial x^i}\left(\frac{\partial \Phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = j^i \tag{18}$$

This is the ith component of the 3d vector equation

$$\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla\left(\nabla\cdot\boldsymbol{A} + \frac{\partial \Phi}{\partial t}\right) = \boldsymbol{j} \tag{19}$$

which agrees with 9. Thus Maxwell's equations with sources in 4d are

$$\boxed{\partial^\alpha \partial_\alpha A^\mu - \partial^\mu\left(\partial_\nu A^\nu\right) = -e j^\mu} \tag{20}$$

PINGBACKS

Pingback: Electromagnetism Lagrangian with sources