MOMENTUM OF A FREE SCALAR KLEIN-GORDON FIELD

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We can work out the 3-momentum of a free scalar Klein-Gordon field in much the same way as we worked out the total Hamiltonian, by integrating the 3-momentum density over the total volume.

The 3-momentum density is given by

$$p^i = \pi_r \frac{\partial \phi^r}{\partial x_i} = -\pi_r \frac{\partial \phi^r}{\partial x^i} \tag{1}$$

where the conjugate momentum is

$$\pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} \tag{2}$$

For the Klein-Gordon free field, the conjugate momentum is

$$\pi = \dot{\phi}^\dagger \tag{3}$$
$$\pi^\dagger = \dot{\phi} \tag{4}$$

Thus the total momentum is given by (recall that there is a sum over r, and that the field and its conjugate are considered as independent fields):

$$P^i = \int p^i d^3x \tag{5}$$
$$= -\int \pi_r \frac{\partial \phi^r}{\partial x^i} d^3x \tag{6}$$
$$= -\int \left[\pi \frac{\partial \phi}{\partial x^i} + \pi^\dagger \frac{\partial \phi^\dagger}{\partial x^i}\right] d^3x \tag{7}$$
$$= -\int \left[\dot{\phi}^\dagger \frac{\partial \phi}{\partial x^i} + \dot{\phi} \frac{\partial \phi^\dagger}{\partial x^i}\right] d^3x \tag{8}$$

The discrete fields are

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} b^{\dagger}(\mathbf{k}) e^{ikx} \tag{9}$$

$$\phi^{\dagger}(x) = \sum_{\mathbf{k}'} \frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}} a^{\dagger}\left(\mathbf{k}'\right) e^{ik'x} + \sum_{\mathbf{k}'} \frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}} b\left(\mathbf{k}'\right) e^{-ik'x} \tag{10}$$

The integrals over x are all of the form $\int e^{ix(k\pm k')}d^3x$ which, because of the condition that an integral number of wavelengths must fit into the volume, works out to zero unless the exponent of the integrand is zero, in which case the integral just gives V. In multiplying out the terms in 8, we get four sums. Consider the term containing products of a and b operators. In this case, since the exponential terms in 9 and 10 are e^{-ikx} and $e^{-ik'x}$, the only non-zero terms in the integral are when $k' = -k$, so we get (recall $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ and $\partial e^{ikx}/\partial x^i = ik_i e^{ikx} = -ik^i e^{ikx}$):

$$P^i_{ab} = \frac{1}{2V}\sum_{\mathbf{k}}\int\left[\frac{-i\omega_{\mathbf{k}}\left(-ik^i\right)}{\omega_{\mathbf{k}}} b(-\mathbf{k})\, a(\mathbf{k}) + \frac{-i\omega_{\mathbf{k}}\left(ik^i\right)}{\omega_{\mathbf{k}}} a(\mathbf{k})\, b(-\mathbf{k})\right] d^3x \tag{11}$$

Since $[a,b] = 0$, the two terms in the integrand cancel and $P^i_{ab} = 0$. The same is true of the term containing products of $a^{\dagger}$ and $b^{\dagger}$, so $P^i_{a^{\dagger}b^{\dagger}} = 0$ and we're left with the terms containing products of $a^{\dagger}$and a, and products of $b^{\dagger}$ and b. In both these cases, since the exponential terms in 9 and 10 are $e^{\pm ikx}$ and $e^{\mp ik'x}$, the only non-zero terms in the integral are when $k' = k$. We get, using $\left[a, a^{\dagger}\right] = \left[b, b^{\dagger}\right] = 1$:

$$P^i_{a^\dagger a} = \frac{1}{2V}\sum_{\mathbf{k}}\int\left[\frac{i\omega_{\mathbf{k}}\left(-ik^i\right)}{\omega_{\mathbf{k}}}a^\dagger\left(\mathbf{k}\right)a\left(\mathbf{k}\right)+\frac{-i\omega_{\mathbf{k}}\left(ik^i\right)}{\omega_{\mathbf{k}}}a\left(\mathbf{k}\right)a^\dagger\left(\mathbf{k}\right)\right]d^3x \tag{12}$$

$$= \sum_{\mathbf{k}}k^i\left[a^\dagger\left(\mathbf{k}\right)a\left(\mathbf{k}\right)+\frac{1}{2}\right] \tag{13}$$

$$= \sum_{\mathbf{k}}k^i\left[N_a\left(\mathbf{k}\right)+\frac{1}{2}\right] \tag{14}$$

$$P^i_{b^\dagger b} = \frac{1}{2V}\sum_{\mathbf{k}}\int\left[\frac{-i\omega_{\mathbf{k}}\left(ik^i\right)}{\omega_{\mathbf{k}}}b\left(\mathbf{k}\right)b^\dagger\left(\mathbf{k}\right)+\frac{i\omega_{\mathbf{k}}\left(-ik^i\right)}{\omega_{\mathbf{k}}}b^\dagger\left(\mathbf{k}\right)b\left(\mathbf{k}\right)\right]d^3x \tag{15}$$

$$= \sum_{\mathbf{k}}k^i\left[b^\dagger\left(\mathbf{k}\right)b\left(\mathbf{k}\right)+\frac{1}{2}\right] \tag{16}$$

$$= \sum_{\mathbf{k}}k^i\left[N_b\left(\mathbf{k}\right)+\frac{1}{2}\right] \tag{17}$$

At first glance, because of the $\frac{1}{2}$ terms in each sum, it looks like the momentum is going to suffer from the same infinity that the total energy does. However, in this case, the sum is over k^i which takes on both positive and negative values, so fortunately here the sums cancel out to zero and we're left with

$$P^i = P^i_{a^\dagger a}+P^i_{b^\dagger b} \tag{18}$$

$$= \sum_{\mathbf{k}}k^i\left[N_a\left(\mathbf{k}\right)+N_b\left(\mathbf{k}\right)\right] \tag{19}$$

Thus the total momentum is just the sum of the momenta of the individual particles (assuming each particle is in an eigenstate $|\phi_{\mathbf{k}}\rangle$). For example, in the state $\left|2\phi_{\mathbf{k}_1},3\bar{\phi}_{\mathbf{k}_1},\bar{\phi}_{\mathbf{k}_2}\right\rangle$, the expectation value is

$$\left\langle 2\phi_{\mathbf{k}_1},3\bar{\phi}_{\mathbf{k}_1},\bar{\phi}_{\mathbf{k}_2}\left|P^i\right|2\phi_{\mathbf{k}_1},3\bar{\phi}_{\mathbf{k}_1},\bar{\phi}_{\mathbf{k}_2}\right\rangle = (2+3)\,k_1^i+k_2^i = 5k_1^i+k_2^i \tag{20}$$