MOMENTUM OF PARTICLES IN A DIRAC FIELD

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To work out the total 3-momentum of the particles (and antiparticles) in a Dirac field, we can start with the 3-momentum density

$$p^i = -\pi_r \frac{\partial \phi^r}{\partial x^i} \tag{1}$$

where the r index is summed, and for the Dirac field, $\phi^1 = \psi$ and $\phi^2 = \bar{\psi}$. The conjugate momenta for the Dirac field are

$$\pi_1 = \pi^{1/2} = i\psi^\dagger \tag{2}$$

$$\pi_2 = \bar{\pi}^{1/2} = 0 \tag{3}$$

The two field solutions are

$$\psi = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ipx} + d_r^\dagger(\mathbf{p})\,v_r(\mathbf{p})\,e^{ipx}\right] \tag{4}$$

$$\bar{\psi} = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,\bar{v}_r(\mathbf{p})\,e^{-ipx} + c_r^\dagger(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right] \tag{5}$$

The latter can be converted into the Hermitian conjugate by post-multiplying the equation by γ^0, which gives

$$\psi^\dagger = \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_r(\mathbf{p})\,v_r^\dagger(\mathbf{p})\,e^{-ipx} + c_r^\dagger(\mathbf{p})\,u_r^\dagger(\mathbf{p})\,e^{ipx}\right] \tag{6}$$

The 3-momentum density is therefore

$$p^{i} = -i\sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_{r}\left(\mathbf{p}\right)v_{r}^{\dagger}\left(\mathbf{p}\right)e^{-ipx}+c_{r}^{\dagger}\left(\mathbf{p}\right)u_{r}^{\dagger}\left(\mathbf{p}\right)e^{ipx}\right]\times \tag{7}$$

$$i\sum_{s=1}^{2}\sum_{\mathbf{q}}\sqrt{\frac{m}{VE_{\mathbf{q}}}}\left[\left(-q_{i}\right)c_{s}\left(\mathbf{q}\right)u_{s}\left(\mathbf{q}\right)e^{-iqx}+q_{i}d_{s}^{\dagger}\left(\mathbf{q}\right)v_{s}\left(\mathbf{q}\right)e^{iqx}\right] \tag{8}$$

$$= \sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left[d_{r}\left(\mathbf{p}\right)v_{r}^{\dagger}\left(\mathbf{p}\right)e^{-ipx}+c_{r}^{\dagger}\left(\mathbf{p}\right)u_{r}^{\dagger}\left(\mathbf{p}\right)e^{ipx}\right]\times \tag{9}$$

$$\sum_{s=1}^{2}\sum_{\mathbf{q}}\sqrt{\frac{m}{VE_{\mathbf{q}}}}\left[q^{i}c_{s}\left(\mathbf{q}\right)u_{s}\left(\mathbf{q}\right)e^{-iqx}-q^{i}d_{s}^{\dagger}\left(\mathbf{q}\right)v_{s}\left(\mathbf{q}\right)e^{iqx}\right] \tag{10}$$

The total momentum is the integral of p^{i} over the volume V:

$$P^{i} = \int p^{i}d^{3}x \tag{11}$$

and, as usual, any terms with an exponential term will integrate to zero due to the requirement that an integral number of wavelengths must fit into the volume. This means that in the product of the first term of 9 with the first term of 10, and in the product of the last term of 9 with the last term of 10, we must have $\mathbf{q} = -\mathbf{p}$. However, due to the orthogonality of the spinors we have

$$v_{r}^{\dagger}\left(\mathbf{p}\right)u_{s}\left(-\mathbf{p}\right) = u_{r}^{\dagger}\left(\mathbf{p}\right)v_{s}\left(-\mathbf{p}\right) = 0 \tag{12}$$

so these terms all equal zero. The remaining terms result from the product of the first term of 9 with the last term of 10, and the product of the last term of 9 with the first term of 10, where in both cases we must have $\mathbf{p} = \mathbf{q}$. Further, because

$$u_{r}^{\dagger}\left(\mathbf{p}\right)u_{s}\left(\mathbf{p}\right) = v_{r}^{\dagger}\left(\mathbf{p}\right)v_{s}\left(\mathbf{p}\right) = \frac{E_{\mathbf{p}}}{m}\delta_{rs} \tag{13}$$

the double sum over spins r and s collapses to a single sum over r. Because $\mathbf{p} = \mathbf{q}$, the exponential terms cancel out and using $\int d^{3}x = V$ we get

$$P^{i} = \sum_{r,\mathbf{p}}p^{i}\left[c_{r}^{\dagger}\left(\mathbf{p}\right)c_{r}\left(\mathbf{p}\right)-d_{r}\left(\mathbf{p}\right)d_{r}^{\dagger}\left(\mathbf{p}\right)\right] \tag{14}$$

Using the anticommutators

$$\left[d_{r}\left(\mathbf{p}\right),d_{s}^{\dagger}\left(\mathbf{p}'\right)\right]_{+} = \delta_{rs}\delta_{\mathbf{pp}'} \tag{15}$$

we get

$$P^{i}=\sum_{r,\mathbf{p}}p^{i}\left[c_{r}^{\dagger}\left(\mathbf{p}\right)c_{r}\left(\mathbf{p}\right)+d_{r}^{\dagger}\left(\mathbf{p}\right)d_{r}\left(\mathbf{p}\right)-1\right] \tag{16}$$

Because the sum over **p** includes both negative and positive momenta, the -1 in the sum cancels out and we're left with

$$P^{i}=\sum_{r,\mathbf{p}}p^{i}\left[c_{r}^{\dagger}\left(\mathbf{p}\right)c_{r}\left(\mathbf{p}\right)+d_{r}^{\dagger}\left(\mathbf{p}\right)d_{r}\left(\mathbf{p}\right)\right] \tag{17}$$

$$=\sum_{r,\mathbf{p}}p^{i}\left(N_{r}\left(\mathbf{p}\right)+\bar{N}_{r}\left(\mathbf{p}\right)\right) \tag{18}$$

For the full vector 3-momentum, we thus have

$$\mathbf{P}=\sum_{r,\mathbf{p}}\mathbf{p}\left(N_{r}\left(\mathbf{p}\right)+\bar{N}_{r}\left(\mathbf{p}\right)\right) \tag{19}$$

That is, the total momentum is just the sum of the momenta over all the particles. For a state $\left|\psi_{r_1\mathbf{p}_1}\psi_{r_2\mathbf{p}_2}\psi_{r_1\mathbf{p}_3}\bar{\psi}_{r_1\mathbf{p}_1}\right\rangle$ the momentum is

$$\mathbf{P}\left|\psi_{r_1\mathbf{p}_1}\psi_{r_2\mathbf{p}_2}\psi_{r_1\mathbf{p}_3}\bar{\psi}_{r_1\mathbf{p}_1}\right\rangle=2\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3 \tag{20}$$

Using a similar derivation, we can find the total charge in a state to be (see Klauber's section 4.7.2)

$$Q=-e\sum_{r,\mathbf{p}}\left(N_{r}\left(\mathbf{p}\right)-\bar{N}_{r}\left(\mathbf{p}\right)\right) \tag{21}$$

where $-e$ is the electron charge. Note that the antiparticle number operator $\bar{N}_{r}\left(\mathbf{p}\right)$ contributes a charge of $+e$ for each antiparticle. Klauber's derivation uses the conserved probability current and integrates it using a method similar to the above.