MOTIVATION FOR A QUANTUM FIELD THEORY

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WHY FIELDS?

We've seen that attempting to create a relativistic quantum theory simply by using the relativistic energy

$$\omega_{\boldsymbol{p}} = \sqrt{p^2 + \mu^2} \tag{1}$$

in the Schrödinger equation leads to the possibility that particles could travel faster than light. In Section 3.1, Coleman discusses the ramifications of this problem, and suggests one way to solve it.

The problem is that, if we have two regions (R_1 and R_2, say) of spacetime that are separated by spacelike intervals, that is, it is impossible for a light ray to travel from any event in R_1 to any event in R_2 (and vice versa), then the probability of a particle passing through one event in R_1 and another event in R_2 should be zero. Another way of saying this is that if we have an observable $\mathscr{O}_1$ that is measurable in R_1 and another observable $\mathscr{O}_2$ measurable in R_2, then these two observables cannot influence each other. In quantum mechanical language, this means that these two observables must commute, so we have

$$[\mathscr{O}_1, \mathscr{O}_2] = 0 \tag{2}$$

for *all* pairs of events with one event in R_1 and the other event in R_2.

At this point, we examine classical physics to see if there is any theory there that satisfies this property. The one area that does have this condition is in Maxwell's theory of electromagnetism. In that theory, we use fields to represent electric and magnetic phenomena. Further, the speed of electromagnetic waves is found to be the speed of light, so any influence of one event on another is limited to that speed. This is just what we need in a relativistic quantum theory.

Using electromagnetic fields as a motivation, it is therefore proposed that relativistic quantum theory should be based on the concept of fields. Ultimately, all that matters in a physical theory is what can be measured, that is, observables, and in quantum theory, observables are represented by hermitian operators. So we propose that relativistic quantum theory is constructed from quantum fields that are represented by operators. Observables are then determined from functions of these fields.

CONDITIONS ON A SCALAR QUANTUM FIELD

The idea is to build a quantum field theory by postulating that a quantum field can be constructed as a linear combination of the creation and annihilation operators we met earlier. That is, for a given field $\phi^a(x)$, we propose that

$$\phi^a(x) = \int d^3\boldsymbol{p}\left[F^a_{\boldsymbol{p}}(x)\,a_{\boldsymbol{p}} + G^a_{\boldsymbol{p}}(x)\,a^\dagger_{\boldsymbol{p}}\right] \tag{3}$$

Here, $F^a_{\boldsymbol{p}}(x)$ and $G^a_{\boldsymbol{p}}(x)$ are numerical functions (not operators) of momentum $\boldsymbol{p}$ and spacetime x. An important point is that the spacetime coordinates are no longer operators; they are just labels. Thus x labels the coordinates, relative to some Lorentz frame, of a particular event in spacetime. The only operators in this definition are $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$.

The index a in $F^a_{\boldsymbol{p}}(x)$ and $G^a_{\boldsymbol{p}}(x)$ is a label for the particular field ϕ^a. It's perhaps an unfortunate choice of notation, being the same symbol as that used for $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$, but as it appears only as a superscript, it shouldn't cause too much confusion. The idea is that we can have a set of quantum fields with each field labelled by an index a, and that various fields all commute with each other, so they are independent.

Coleman imposes 4 conditions on the fields ϕ^a. First, we must have

$$\left[\phi^a(x), \phi^b(y)\right] = 0 \tag{4}$$

if $(x-y)^2 < 0$. This is the condition described above, that all pairs of observables separated by a spacelike interval must be independent of each other to preserve causality and the condition that nothing can travel faster than light.

The second condition is that the fields $\phi^a(x)$ are Hermitian, since the fields are to be observable. That is, we must have

$$\phi^a(x) = \phi^a(x)^\dagger \tag{5}$$

The third and fourth conditions are that the fields must transform properly under translations and Lorentz transformations. We've considered translations in non-relativistic quantum mechanics, and the same argument applies here. To translate a state $|\psi\rangle$ by a position vector $\boldsymbol{a}$ (argh - another use of the symbol 'a'. Be careful here!), then the transformed state is

$$\left|\psi'\right\rangle = e^{-i\boldsymbol{P}\cdot\boldsymbol{a}}\left|\psi\right\rangle \tag{6}$$

where $\boldsymbol{P}$ is the 3-momentum operator. As we saw earlier, if we consider the expectation value of some quantity $\rho(\boldsymbol{x})$, that is

$$f(\boldsymbol{x}) = \langle\psi\left|\rho(\boldsymbol{x})\right|\psi\rangle \tag{7}$$

then if we translate the system bodily by a distance $\boldsymbol{a}$ so it is now $|\psi'\rangle$, then the expectation value

$$f'(\boldsymbol{x}) = \left\langle\psi'\left|\rho(\boldsymbol{x})\right|\psi'\right\rangle \tag{8}$$

must be

$$f'(\boldsymbol{x}) = f(\boldsymbol{x}-\boldsymbol{a}) \tag{9}$$

Therefore

$$\left\langle\psi'\left|\rho(\boldsymbol{x})\right|\psi'\right\rangle = \left\langle\psi\left|e^{i\boldsymbol{P}\cdot\boldsymbol{a}}\rho(\boldsymbol{x})\,e^{-i\boldsymbol{P}\cdot\boldsymbol{a}}\right|\psi\right\rangle \tag{10}$$

so from 9 we must have

$$f(\boldsymbol{x}-\boldsymbol{a}) = \left\langle\psi\left|e^{i\boldsymbol{P}\cdot\boldsymbol{a}}\rho(\boldsymbol{x})\,e^{-i\boldsymbol{P}\cdot\boldsymbol{a}}\right|\psi\right\rangle \tag{11}$$

Comparing this with 7, we have

$$\rho(\boldsymbol{x}-\boldsymbol{a}) = e^{i\boldsymbol{P}\cdot\boldsymbol{a}}\rho(\boldsymbol{x})\,e^{-i\boldsymbol{P}\cdot\boldsymbol{a}} \tag{12}$$

This condition was derived in 3-space, but we can generalize it to space-time and write, for a field $\phi(x)$:

$$\boxed{\phi(x-a) = e^{-iP\cdot a}\phi(x)\,e^{iP\cdot a}} \tag{13}$$

Coleman deals with Lorentz invariance by first considering the special case of rotations, since a rotation is a special case of a Lorentz transformation. For a scalar field ρ, we get a condition analogous to 13:

$$U(R)^{\dagger}\rho(\boldsymbol{x})\,U(R) = \rho\left(R^{-1}\boldsymbol{x}\right) \tag{14}$$

That is, if we rotate the entire system $|\psi\rangle$ by applying the rotation matrix R, then the value of ρ at the rotated position is found by looking at the value of ρ at the original location, before the rotation. This is generalized to an arbitrary Lorentz transformation Λ so we have

$$\boxed{\phi\left(x'\right) = U(\Lambda)^{\dagger}\phi(x)\,U(\Lambda) = \phi\left(\Lambda^{-1}x\right)} \tag{15}$$

The first two conditions are physically vital, since they ensure, first, compatibility with causality in special relativity, and, second, that fields are observable. The conditions concerning translations and Lorentz transformations are, Coleman says, 'just simplifying assumptions'. I'm assuming what he means by this is that we're restricting out attention to scalar fields for the moment. When we consider more complex objects, we will have to modify these conditions.

The condition that the scalar fields $\phi^{a}(x)$ are linear combinations of the creation and annihilation operators is, at the moment, a simplifying assumption, but it will turn out to give us a valid theory.

PINGBACKS

Pingback: Lorentz transformation and translation of explicit quantum field

Pingback: Scalar fields - summary