NOETHER'S THEOREM AND LORENTZ TRANSFORMATIONS

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Post date: 11 August 2021.

Coleman's treatment of the conservation laws obtained from Lorentz invariance is considerably easier to follow than most other treatments I've seen, so it's worth summarizing it here.

We consider an infinitesimal Lorentz transformation Λ, under which a spacetime vector transforms as

$$x^{\mu}\to x^{\mu}+\epsilon^{\mu\nu}x_{\nu}d\lambda \tag{1}$$

where $\epsilon^{\mu\nu}$ is a 4×4 matrix and $d\lambda$ is an infinitesimal quantity. We then consider the transformation on another spacetime vector y_{μ}, this time with lower indices, so we have

$$y_{\mu}=y_{\mu}+\epsilon_{\mu\nu}y^{\nu}d\lambda \tag{2}$$

Because the scalar product $x^{\mu}y_{\mu}$ is a Lorentz invariant, the scalar product of the transformed vectors must be the same as before the transformation. This gives rise to Coleman's eqn 5.62, where, to first order in $d\lambda$ we have

$$x^{\mu}y_{\mu}\to x^{\mu}y_{\mu}+\left(\epsilon_{\mu\nu}+\epsilon_{\nu\mu}\right)x^{\mu}y^{\nu}d\lambda \tag{3}$$

Note that $\epsilon_{\mu\nu}$ is not necessarily an infinitesimal. The only infinitesimal in this transformation is in the parameter $d\lambda$. The quantities $\epsilon_{\mu\nu}$ are constants for any given Lorentz transformation.

Lorentz invariance then requires that the RHS is the same as the LHS, so

$$\epsilon_{\mu\nu}=-\epsilon_{\nu\mu} \tag{4}$$

Thus $\epsilon_{\mu\nu}$ is an antisymmetric matrix. We could run through the same derivation with all upper indices lowered and vice versa, so we would also get the condition

$$\epsilon^{\mu\nu}=-\epsilon^{\nu\mu} \tag{5}$$

However, it's important to note that if we considered the matrix with one upper and one lower index, such as ϵ_{ν}^{μ}, this is not necessarily antisymmetric; it probably doesn't have any symmetry at all.

Thus a general Lorentz transformation has 6 independent parameters, since a 4×4 antisymmetric matrix has 6 independent elements (say, the elements in the triangle above the diagonal). These correspond to the 3 independent parameters in a rotation, and the 3 independent parameters in a Lorentz boost (a constant velocity between two inertial frames).

Coleman then gives a couple of examples of specific Lorentz transformations, one for a rotation (with $\epsilon^{12}=1=-\epsilon^{21}$ and all other entries zero) and one for a boost (with $\epsilon^{10}=1=-\epsilon^{01}$). The first example gives a standard rotation about the x^3 axis; the second gives a boost along the x^1 axis and gives the standard Lorentz transformation equations encountered in an introduction to special relativity.

What concerns us here is the application of Noether's theorem to a field theory system that exhibits Lorentz symmetry. As with translational symmetry, we wish to find the current given by

$$J^\mu \equiv \pi_a^\mu D\phi^a - F^\mu \tag{6}$$

With translational symmetry, we had a single vector e_ρ that specified the direction of the translation and required that the currents J^μ be linear combinations of the components e_ρ. In the Lorentz case, we have a 4×4 matrix $\epsilon_{\lambda\rho}$ characterizing the transformation, so we require the currents to be linear combinations of the elements of that matrix. The most general such linear combination is

$$J^\mu = \frac{1}{2}\epsilon_{\lambda\rho}M^{\lambda\rho\mu} \tag{7}$$

where $M^{\lambda\rho\mu}$ is some general quantity with $4^3=64$ components. However, any matrix can be written as the sum of a symmetric and antisymmetric matrix. Consider, for example, a square matrix $Q^{\alpha\beta}$. We can write this as

$$Q = \frac{1}{2}\left(Q+Q^T\right)+\frac{1}{2}\left(Q-Q^T\right) \tag{8}$$

The first term on the RHS is symmetric and the second term in antisymmetric.

Because the product $\epsilon_{\lambda\rho}S^{\lambda\rho}=0$ for any antisymmetric matrix ϵ and symmetric matrix S, we can ignore the symmetric part of $M^{\lambda\rho\mu}$and define $M^{\lambda\rho\mu}$ in 7 to be antisymmetric in the indices $\lambda\rho$.

Because the current satisfies the conservation law

$$\partial_\mu J^\mu = 0 \tag{9}$$

and because the $\epsilon_{\lambda\rho}$ matrix is arbitrary (up to the antisymmetry condition) we therefore have the condition on $M^{\lambda\rho\mu}$

$$\partial_\mu M^{\lambda\rho\mu} = 0 \tag{10}$$

As we've seen earlier, the component J^0 represents some 'stuff' that is conserved and the spatial components $\boldsymbol{J}$ represent the current of the J^0 'stuff'. In this case

$$J^0 = \frac{1}{2}\epsilon_{\lambda\rho}M^{\lambda\rho 0} \tag{11}$$

Again, because the $\epsilon_{\lambda\rho}$ matrix is arbitrary, we can integrate the $M^{\lambda\rho 0}$ component over all space to get the total amount of 'stuff' that is conserved. Rather confusingly, Coleman uses the same symbol J but now with *two* indices to represent this stuff:

$$J^{\lambda\rho} = \int d^3\boldsymbol{x}\; M^{\lambda\rho 0} \tag{12}$$

When reading the rest of this section in Coleman's book, you need to remember that $J^{\lambda\rho}$ is *not* a component of the Noether current; it's a separate conserved quantity. In fact, when both λ and ρ take on values from $(1,2,3)$, we get the 3 components of angular momentum. This comes out of the rest of the section in Coleman's book.

To get this result, we need to evaluate 6 for an infinitesimal Lorentz transformation. We consider a Lorentz transformation Λ of the field $\phi^a(x)$. As usual with these tranformations, transforming a field is equivalent to looking at the value the field had in the old location, so we have

$$\phi^a(x) \to \phi^a\left(\Lambda^{-1}x\right) \tag{13}$$

(See here for a discussion of this point, particularly eqn 19.) Coleman shows this results in

$$D\phi^a = \left.\frac{\partial\phi^a}{\partial\lambda}\right|_{\lambda=0} = \epsilon_{\sigma\rho}x^\sigma\partial^\rho\phi^a \tag{14}$$

Although this was derived specifically for a field $\phi^a(x)$, the derivation in fact applies to any function that is a Lorentz scalar. Since we're assuming the Lagrangian $\mathscr{L}$ is also a Lorentz scalar, we can write the same equation for $\mathscr{L}$:

$$D\mathscr{L} = \epsilon_{\sigma\rho}x^\sigma\partial^\rho\mathscr{L} \tag{15}$$

This is converted to a divergence so that we can obtain the object F^μ in 6, and we get

$$\partial_\mu F^\mu = \partial_\mu\left(g^{\mu\rho}\epsilon_{\sigma\rho}x^\sigma\mathscr{L}\right) \tag{16}$$

This gives the final result that the current is

$$J^\mu = \pi^\mu_a D\phi^a - F^\mu \tag{17}$$
$$= \epsilon_{\sigma\rho} x^\sigma \left(\pi^\mu_a \partial^\rho \phi^a - g^{\mu\rho} \mathscr{L}\right) \tag{18}$$

The quantity in parentheses is the same as the energy-momentum tensor $T^{\rho\mu}$ we obtained when considering translational symmetry. Therefore we have

$$J^\mu = \epsilon_{\sigma\rho} x^\sigma T^{\rho\mu} \tag{19}$$

Again, since $\epsilon_{\sigma\rho}$ is antisymmetric, we can write

$$x^\sigma T^{\rho\mu} = \frac{1}{2}\left(x^\sigma T^{\rho\mu} + x^\rho T^{\sigma\mu}\right) + \frac{1}{2}\left(x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu}\right) \tag{20}$$

and observe that the sum over the first term (the term symmetric in σ and ρ) with $\epsilon_{\sigma\rho}$ gives zero, so we're left with

$$J^\mu = \frac{1}{2}\epsilon_{\sigma\rho}\left(x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu}\right) \tag{21}$$

Comparing with 7 we get

$$M^{\sigma\rho\mu} = x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu} \tag{22}$$

In the remainder of the section, Coleman shows that the three independent quantitiesM^{ij0} are densities of the three components of angular momentum. The other three components M^{i00}, when integrated, simply expresses the fact that the centre of mass (or, more properly in relativity, the centre of energy) has a constant velocity.